A tale of two complete positivities

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Abstract

In physics, one has the notion of a completely positive map from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{m\times m}$ in the sense of Stinespring or Choi. The set of all such maps happens to form a cone. In optimization, one also has a cone of real "completely positive matrices," studied by Berman and Shaked-Monderer among others. We investigate how to the two concepts are related.

1 Introduction

Man-Duen Choi famously characterized the completely positive complex-linear maps from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{m\times m}$. According to Choi [3],

Definition 1 (Choi CP v1)**.** A complex-linear map $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is completely positive iff for all $p \in \mathbb{N}$ and all $A_{ij} \in \mathbb{C}^{n\times n}$, we have

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & & & \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix} \succcurlyeq 0 \implies \begin{bmatrix} \Phi(A_{11}) & \Phi(A_{12}) & \cdots & \Phi(A_{1p}) \\ \Phi(A_{21}) & \Phi(A_{22}) & \cdots & \Phi(A_{2p}) \\ \vdots & & & \\ \Phi(A_{p1}) & \Phi(A_{p2}) & \cdots & \Phi(A_{pp}) \end{bmatrix} \succcurlyeq 0.$$

Choi's theorem[1] characterizes the set of all such maps,

$$\mathbf{CP}_{n,m}(\mathbb{C}) = \left\{ A \mapsto \sum_{i=1}^{k} V_i A V_i^* \;\middle|\; V_i \in \mathbb{C}^{m\times n}, k \in \mathbb{N} \right\}.$$

Here, $X \succcurlyeq 0$ means that $X \in \mathbb{C}^{n\times n}$ is positive-semidefinite (PSD), and a complex PSD matrix (acting on $\mathbb{C}^n$) is necessarily Hermitian. $\mathbb{C}^{n\times n}$ is an example of a C^* algebra, and in that regard, Choi's theorem is a special case of an earlier result known as Stinespring's dilation theorem [9].

It is fairly easy to see that $\mathbf{CP}_{n,m}(\mathbb{C})$ forms a cone. In fact, Barker, Hill, and Haertel later showed [1] that it is isomorphic to the PSD cone $\mathcal{H}_+^{mn}(\mathbb{C})$. This isomorphism maps the ambient space of Hermitian-preserving maps to the space $\mathcal{H}^{mn}(\mathbb{C})$ of Hermitian mn-by-mn matrices wherein $\mathcal{H}_+^{mn}(\mathbb{C})$ is self-dual.

[1] https://en.wikipedia.org/wiki/Choi's_theorem_on_completely_positive_maps

From this we conclude that $\mathbf{CP}_{n,m}(\mathbb{C})$ is self-dual (in fact, symmetric) in the ambient space of Hermitian-preserving maps from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{m\times m}$.

In optimization over the reals, there is a competing notion of complete positivity [8]. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be completely-positive if it can be written as $A = BB^T$ where $B \in \mathbb{R}^{n\times n}$ is entrywise nonnegative. If $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, then this is equivalent to saying that $A = \sum_{i=1}^n x_i x_i^T$ for $x_i \in \mathbb{R}^n_+$. The completely-positive matrices also form a cone, and are important because they can be used in cone programs to solve hard problems [2].

The completely positive matrices were generalized by Sturm and Zhang [10] to the set of transformations that is completely-positive on a set K.

Definition 2. The *completely positive* cone of $K \subseteq \mathbb{R}^n$ is

$$\operatorname{cone}\left(\left\{xx^T \;\middle|\; x \in K\right\}\right).$$

The cone of completely-positive matrices is then recovered by taking $K := \mathbb{R}^n_+$. It is the dual of the *set-semidefinite* cone, or generalized copositive cone, and both have been extensively studied [4, 5].

Question 1. Does Choi's characterization extend to real-linear maps on real matrices?

Question 2. If the answer to the first question is yes, then do the two definitions of complete positivity coincide? Or at least, how are they related?

We're not the first to wonder about this connection. A recent survey paper on Choi representations remarks [6], "it should be noted that 'completely positivity' of linear operators can have another meaning: especially in the literature of mathematical optimization... with implications to quadratic and combinatorial optimization. However, there is no connection known between this latter notion of 'complete positivity' and the one discussed in the present tutorial." There is also a Reddit thread discussing bad math notation[2] where the top-rated comment complains that completely-positive maps and completely-positive matrices are unrelated.

The first step towards answering these questions is to understand how the Kronecker product is used in Definition 1. This will allow us to undo the "box of numbers" representation and put everything in terms of linear operators. (We technically don't need to understand Choi's original definition at all—only his characterizaion—but the tools we develop in the process of the latter get us most of the way to the former. So *while we're here...*)

2 The setting

- V and W are finite-dimensional inner-product spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

[2] https://www.reddit.com/r/math/comments/41r4sg/

- $\mathbf{e} = \{e_1, e_2, \ldots, e_n\}$ and $\mathbf{f} = \{f_1, f_2, \ldots, f_m\}$ are orthonormal bases for V and W, respectively.
- If L is linear on V and if $x \in V$, we write $\mathbf{e}(L)$, a matrix, and $\mathbf{e}(x)$, a vector, for the basis representations of L and x. For any $x \in V$ and linear $L : V \to V$, we have $\mathbf{e}(L(x)) = \mathbf{e}(L)\,\mathbf{e}(x)$ in $\mathbb{F}^n$. Vectors are column.
- If $x \in V$ and $s \in W$, we define $|x\rangle\langle s| : W \to V$ to be the linear map $z \mapsto \langle s, z\rangle\, x$. (This is a *ket-bra*, based on Dirac's bra-ket notation.[3]) Typographically this notation becomes pretty horrific, but it's standard.
- $\mathcal{B}(W, V)$ is the set of all $\mathbb{F}$-linear operators from V to W having orthonormal basis

$$\mathbf{g} = \begin{Bmatrix} |e_1\rangle\langle f_1|\,, & |e_1\rangle\langle f_2|\,, & \ldots, & |e_1\rangle\langle f_m|\,, \\ |e_2\rangle\langle f_1|\,, & |e_2\rangle\langle f_2|\,, & \ldots, & |e_2\rangle\langle f_m|\,, \\ & & \vdots & \\ |e_n\rangle\langle f_1|\,, & |e_n\rangle\langle f_2|\,, & \ldots, & |e_n\rangle\langle f_m| \end{Bmatrix},$$

  ordered lexicographically (left to right then top to bottom), and the trace inner product defined by setting $\langle |x\rangle\langle s|, |y\rangle\langle z|\rangle \coloneqq \langle x, y\rangle_V \langle s, z\rangle_W$ and extending via bilinearity. We abbreviate $\mathcal{B}(V, V)$ as $\mathcal{B}(V)$.
- If $A \in \mathcal{B}(V)$ and $B \in \mathcal{B}(W)$, then the map $A \odot B \in \mathcal{B}(\mathcal{B}(W, V))$ is defined as sending $|e_i\rangle\langle f_j|$ to $|A(e_i)\rangle\langle B(f_j)|$ and then extended using linearity.
- If A, B are n-by-n and m-by-m matrices, then

$$A \odot_k B \coloneqq \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ & & \vdots & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

  is the Kronecker product [7] of A and B.
- $E_{ij}^{(p)}$ denotes the $p \times p$ matrix with a 1 in the i, jth position and zeros elsewhere. To accomodate the corresponding linear operators, we define one last family of vector spaces $Q^{(p)}$ with orthonormal basis $\mathbf{q}^{(p)} \coloneqq \left\{q_1^{(p)}, q_2^{(p)}, \ldots, q_p^{(p)}\right\}$ so that $\dim\left(Q^{(p)}\right) = p$ and $\left|q_i^{(p)}\right\rangle\left\langle q_j^{(p)}\right| \in \mathcal{B}\left(Q^{(p)}\right)$.

3 Understanding the Kronecker product

We begin by recalling Choi's definition of complete positivity from Definition 1. Consider the matrix $E_{22}^{(p)}$ which is zero except for a 1 in the second row/column.

[3] https://en.wikipedia.org/wiki/Bra-ket_notation

If we take its Kronecker product with an arbitrary matrix A, then

$$E_{22}^{(p)} \odot_k A = \begin{bmatrix} 0A & 0A & \cdots & 0A \\ 0A & 1A & \cdots & 0A \\ & & \vdots & \\ 0A & 0A & \cdots & 0A \end{bmatrix}.$$

In other words, it puts A in the $(2,2)$ position of a p-by-p block matrix. "Clearly" the same thing happens for any (i,j) and not just $(2,2)$. Using this we can rewrite Choi's definition in terms of the Kronecker product.

Definition 3 (Choi CP v2)**.** A complex-linear map $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is completely positive iff for all $p \in \mathbb{N}$ and all $A_{ij} \in \mathbb{C}^{n\times n}$, we have

$$\left[\sum_{i,j=1}^{p} E_{ij}^{(p)} \odot_k A_{ij}\right] \succcurlyeq 0 \implies \left[\sum_{i,j=1}^{p} E_{ij}^{(p)} \odot_k \Phi\left(A_{ij}\right)\right] \succcurlyeq 0.$$

To understand this in terms of linear operators, we apparently must understand the Kronecker product of matrices. First we recall how linear transformations are represented in terms of a basis. For $x \in V$, we have,

$$\mathbf{e}\left(x\right) = \begin{bmatrix} \langle e_1, x\rangle \\ \langle e_2, x\rangle \\ \vdots \\ \langle e_n, x\rangle \end{bmatrix}.$$

Therefore for $L \in \mathcal{B}\left(V\right)$,

$$\begin{aligned} \mathbf{e}\left(L\right) &= \begin{bmatrix} \mathbf{e}\left(L\left(e_1\right)\right) & \mathbf{e}\left(L\left(e_2\right)\right) & \cdots & \mathbf{e}\left(L\left(e_n\right)\right) \end{bmatrix} \\ &= \begin{bmatrix} \langle e_1, L\left(e_1\right)\rangle & \langle e_1, L\left(e_2\right)\rangle & \cdots & \langle e_1, L\left(e_n\right)\rangle \\ \langle e_2, L\left(e_1\right)\rangle & \langle e_2, L\left(e_2\right)\rangle & \cdots & \langle e_2, L\left(e_n\right)\rangle \\ & & \vdots & \\ \langle e_n, L\left(e_1\right)\rangle & \langle e_n, L\left(e_2\right)\rangle & \cdots & \langle e_n, L\left(e_n\right)\rangle \end{bmatrix}. \end{aligned}$$

Proposition 1. *If A, B are linear on V, W, then*

$$\mathbf{g}\left(A \odot B\right) = \mathbf{e}\left(A\right) \odot_k \mathbf{f}\left(B\right).$$

Proof. You just have to do it. Abbreviating $|e_i\rangle\langle f_j|$ by g_{ij},

$$\mathbf{g}\left(A \odot B\right)$$
$$=$$
$$\begin{bmatrix} \langle g_{11}, A \odot B\left(g_{11}\right)\rangle & \cdots & \langle g_{11}, A \odot B\left(g_{1m}\right)\rangle & \cdots & \langle g_{11}, A \odot B\left(g_{nm}\right)\rangle \\ \langle g_{12}, A \odot B\left(g_{11}\right)\rangle & \cdots & \langle g_{12}, A \odot B\left(g_{1m}\right)\rangle & \cdots & \langle g_{12}, A \odot B\left(g_{nm}\right)\rangle \\ & & \vdots & & \\ \langle g_{1m}, A \odot B\left(g_{11}\right)\rangle & \cdots & \langle g_{1m}, A \odot B\left(g_{1m}\right)\rangle & \cdots & \langle g_{1m}, A \odot B\left(g_{nm}\right)\rangle \\ \langle g_{21}, A \odot B\left(g_{11}\right)\rangle & \cdots & \langle g_{21}, A \odot B\left(g_{1m}\right)\rangle & \cdots & \langle g_{21}, A \odot B\left(g_{nm}\right)\rangle \\ & & \vdots & & \\ \langle g_{nm}, A \odot B\left(g_{11}\right)\rangle & \cdots & \langle g_{nm}, A \odot B\left(g_{1m}\right)\rangle & \cdots & \langle g_{nm}, A \odot B\left(g_{nm}\right)\rangle \end{bmatrix}$$

In general, $\langle g_{ij}, A \odot B\left(g_{k\ell}\right)\rangle = \langle e_i, A\left(e_k\right)\rangle \langle f_j, B\left(f_\ell\right)\rangle$. So, consider the top-left $m \times m$ block:

$$\begin{aligned} \mathbf{g}\left(A \odot B\right)_{1:m,1:m} &= \begin{bmatrix} \langle e_1, A\left(e_1\right)\rangle \langle f_1, B\left(f_1\right)\rangle & \cdots & \langle e_1, A\left(e_1\right)\rangle \langle f_1, B\left(f_m\right)\rangle \\ \langle e_1, A\left(e_1\right)\rangle \langle f_2, B\left(f_1\right)\rangle & \cdots & \langle e_1, A\left(e_1\right)\rangle \langle f_2, B\left(f_m\right)\rangle \\ & \vdots & \\ \langle e_1, A\left(e_1\right)\rangle \langle f_m, B\left(f_1\right)\rangle & \cdots & \langle e_1, A\left(e_1\right)\rangle \langle f_m, B\left(f_m\right)\rangle \end{bmatrix} \\ &= \mathbf{e}\left(A\right)_{11} \begin{bmatrix} \langle f_1, B\left(f_1\right)\rangle & \langle f_1, B\left(f_2\right)\rangle & \cdots & \langle f_1, B\left(f_m\right)\rangle \\ \langle f_2, B\left(f_1\right)\rangle & \langle f_2, B\left(f_2\right)\rangle & \cdots & \langle f_2, B\left(f_m\right)\rangle \\ & & \vdots & \\ \langle f_m, B\left(f_1\right)\rangle & \langle f_m, B\left(f_2\right)\rangle & \cdots & \langle f_m, B\left(f_m\right)\rangle \end{bmatrix} \\ &= \mathbf{e}\left(A\right)_{11} \mathbf{f}\left(B\right). \end{aligned}$$

Similarly, the i, jth $m \times m$ block will be $\mathbf{e}\left(A\right)_{ij} \mathbf{f}\left(B\right)$. In other words, the entire thing is $\mathbf{e}\left(A\right) \odot_k \mathbf{f}\left(B\right)$. □

Proposition 2. *If $x \in V$ and $s \in W$, then $\mathbf{g}\left(|x\rangle\langle s|\right) = \mathbf{e}\left(x\right) \odot_k \mathbf{f}\left(s\right) \in \mathbb{F}^{nm}$.*

Proof. By definition, we know the $\mathbf{g}$-coordinates of $|x\rangle\langle s|$ are,

$$\mathbf{g}\left(|x\rangle\langle s|\right) = \begin{bmatrix} \langle |e_1\rangle\langle f_1|, |x\rangle\langle s|\rangle \\ \langle |e_1\rangle\langle f_2|, |x\rangle\langle s|\rangle \\ \vdots \\ \langle |e_1\rangle\langle f_m|, |x\rangle\langle s|\rangle \\ \langle |e_2\rangle\langle f_1|, |x\rangle\langle s|\rangle \\ \langle |e_2\rangle\langle f_2|, |x\rangle\langle s|\rangle \\ \vdots \\ \langle |e_n\rangle\langle f_m|, |x\rangle\langle s|\rangle \end{bmatrix} = \begin{bmatrix} \langle e_1, x\rangle \langle f_1, s\rangle \\ \langle e_1, x\rangle \langle f_2, s\rangle \\ \vdots \\ \langle e_1, x\rangle \langle f_m, s\rangle \\ \langle e_2, x\rangle \langle f_1, s\rangle \\ \langle e_2, x\rangle \langle f_2, s\rangle \\ \vdots \\ \langle e_n, x\rangle \langle f_m, s\rangle \end{bmatrix} = \mathbf{e}\left(x\right) \odot_k \mathbf{f}\left(s\right).$$ □

Corollary 1. $\mathbf{g}\left(|A\left(x\right)\rangle\langle B\left(s\right)|\right) = \left[\mathbf{e}\left(A\right) \odot_k \mathbf{f}\left(B\right)\right]\left[\mathbf{e}\left(x\right) \odot_k \mathbf{f}\left(s\right)\right]$.

Remark 1. The same result holds if $\mathcal{B}(W, V)$ is replaced by $V \otimes W$, the abstract tensor product [7] of V and W. So in essence, the Kronecker product gives us a basis representation on a tensor product space in terms of the bases for its constituent spaces. (We'll encounter the tensor product again later, but a precise definition isn't important right now.)

Now's a good time to recall what we are trying to accomplish. In Definition 3, we have an expression in terms of,

1. Complex n-by-n matrices A_{ij},
2. Complex m-by-m matrices $\Phi(A_{ij})$,
3. Complex n-by-n matrices $E_{ij}^{(p)}$,
4. The Kronecker product.

The goal, more or less, is to get rid of the matrices and replace them with linear operators. The matrices $A_{ij} \in \mathbb{C}^{n\times n}$ are in one-to-one correspondence with the linear transformations $\mathcal{A}_{ij} \in \mathcal{B}(V)$ under any basis representation map, and in particular under $\mathbf{e}$. Similarly, the matrices $E_{ij}^{(p)} \in \mathbb{C}^{p\times p}$ represent an element of $\mathcal{B}\left(Q_{(p)}\right)$ while the $\Phi(A_{ij})$ represent elements of $\mathcal{B}(W)$.

Lemma 1. *If $x, s \in V$, then $\mathbf{e}(|x\rangle\langle s|) = \mathbf{e}(x)\,\mathbf{e}(s)^*$. In particular,*

$$\mathbf{e}(|e_i\rangle\langle e_j|) = \mathbf{e}(e_i)\,\mathbf{e}(e_j)^* = E_{ij}^{(n)}.$$

Proof. Keeping in mind that the inner product is *sesquilinear*,

$$\begin{aligned}
\mathbf{e}(|x\rangle\langle s|) &= \begin{bmatrix}\mathbf{e}(|x\rangle\langle s|(e_1)) & \mathbf{e}(|x\rangle\langle s|(e_2)) & \cdots & \mathbf{e}(|x\rangle\langle s|(e_n))\end{bmatrix} \\
&= \begin{bmatrix}\langle s, e_1\rangle\,\mathbf{e}(x) & \langle s, e_2\rangle\,\mathbf{e}(x) & \cdots & \langle s, e_n\rangle\,\mathbf{e}(x)\end{bmatrix} \\
&= \begin{bmatrix}\overline{\mathbf{e}(s)_1}\,\mathbf{e}(x) & \overline{\mathbf{e}(s)_2}\,\mathbf{e}(x) & \cdots & \overline{\mathbf{e}(s)_n}\,\mathbf{e}(x)\end{bmatrix} \\
&= \mathbf{e}(x)\,\mathbf{e}(s)^*.
\end{aligned}$$

□

We now have pretty much everything we need to understand (Choi) complete positivity in terms of linear operators.

4 Complete positivity for complex-linear maps

Theorem 1. *The complex-linear map $\Phi : \mathcal{B}(V) \to \mathcal{B}(W)$ is completely positive iff for all $p \in \mathbb{N}$ and all $\mathcal{A}_{ij} \in \mathcal{B}(V)$, we have*

$$\left[\sum_{i,j=1}^{p}\left(\left|q_i^{(p)}\right\rangle\!\left\langle q_j^{(p)}\right|\right)\odot\mathcal{A}_{ij}\right] \succcurlyeq 0 \implies \left[\sum_{i,j=1}^{p}\left(\left|q_i^{(p)}\right\rangle\!\left\langle q_j^{(p)}\right|\right)\odot\Phi(\mathcal{A}_{ij})\right] \succcurlyeq 0.$$

Proof. There's not much left to prove here. Start with Definition 3, and replace the matrices with $\mathbf{e}$, $\mathbf{f}$, and $\mathbf{q}^{(p)}$ representations:

$$\forall p \in \mathbb{N},\ A_{ij} \in \mathbb{C}^{n\times n}$$

$$\left[\sum_{i,j=1}^{p} E_{ij}^{(p)} \odot_k A_{ij}\right] \succcurlyeq 0 \implies \left[\sum_{i,j=1}^{p} E_{ij}^{(p)} \odot_k \Phi\left(A_{ij}\right)\right] \succcurlyeq 0$$

$$\Updownarrow$$

$$\forall p \in \mathbb{N},\ \mathcal{A}_{ij} \in \mathcal{B}\left(V\right)$$

$$\left[\sum_{i,j=1}^{p} \mathbf{q}^{(p)}\left(\left|q_i^{(p)}\middle\rangle\middle\langle q_j^{(p)}\right|\right) \odot_k \mathbf{e}\left(\mathcal{A}_{ij}\right)\right] \succcurlyeq 0$$

$$\implies$$

$$\left[\sum_{i,j=1}^{p} \mathbf{q}^{(p)}\left(\left|q_i^{(p)}\middle\rangle\middle\langle q_j^{(p)}\right|\right) \odot_k \mathbf{f}\left(\Phi\left(\mathcal{A}_{ij}\right)\right)\right] \succcurlyeq 0.$$

Finally, undo the basis representations using the results of the previous section. □

Theorem 2. *Choi's characterization also extends to complex-linear operators,*

$$\mathbf{CP}_{V,W}\left(\mathbb{C}\right) = \operatorname{cone}\left(\left\{\mathcal{U} \odot \mathcal{U} \mid \mathcal{U} \in \mathcal{B}\left(V, W\right)\right\}\right).$$

Proof. Suppose $\Phi = A \mapsto UAU^*$ on $\mathbb{C}^{n\times n}$ for some $U \in \mathbb{C}^{m\times n}$. If the matrix A has rank r, then it can be expressed as the sum of r rank-one matrices $A = \sum_{i=1}^{r} x_i y_i^*$ by using the singular value decomposition and by absorbing the singular values into the vectors. As a result,

$$\Phi\left(A\right) = \sum_{i=1}^{r} Ux_i \left(Uy_i\right)^*$$

We may now suppose that all of the matrices and vectors involved are basis representations,

$$\mathbf{f}\left(\Phi\left(A\right)\right) = \sum_{i=1}^{r} \mathbf{f}\left(\mathcal{U}\left(x_i\right)\right) \mathbf{f}\left(\mathcal{U}\left(y_i\right)\right)^* = \sum_{i=1}^{r} \mathbf{f}\left(\left|\mathcal{U}\left(x_i\right)\right\rangle\!\left\langle\mathcal{U}\left(y_i\right)\right|\right)$$

and undo that basis representation,

$$\Phi\left(\mathcal{A}\right) = \sum_{i=1}^{r} \left|\mathcal{U}\left(x_i\right)\right\rangle\!\left\langle\mathcal{U}\left(y_i\right)\right| = \mathcal{U} \odot \mathcal{U}\left(\sum_{i=1}^{r} \left|x_i\right\rangle\!\left\langle y_i\right|\right) = \mathcal{U} \odot \mathcal{U}\left(\mathcal{A}\right).$$

In other words, $\Phi = \mathcal{U} \odot \mathcal{U}$. Taking sums of such things gives the desired result. □

5 Complete positivity for real-linear maps

Choi's characterization works for real symmetric matrices over $\mathbb{R}$, too.

Theorem 3. *If we replace $\mathbb{C}$ by $\mathbb{R}$ in the definition, then the set of completely-positive real-linear maps from $\mathcal{B}(V)$ to $\mathcal{B}(W)$ is,*

$$\mathbf{CP}_{V,W}(\mathbb{R}) = \operatorname{cone}\left(\{\mathcal{U} \odot \mathcal{U} \mid \mathcal{U} \in \mathcal{B}(V, W)\}\right). \tag{1}$$

Proof. There's nothing in Choi's proof that works for $\mathbb{C}$ but not $\mathbb{R}$. □

That answers Question 1. But can we use the name "completely positive" for such a transformation? Remember, we already have a Definition 2 for the completely-positive cone (of matrices) in $\mathbb{R}^n$. If we write that definition in terms of real-linear operators using the results from Section 3, we get...

Definition 4. The *completely positive* cone of $K \subseteq X$ is

$$\operatorname{cone}\left(\{|x\rangle\langle x| \mid x \in K\}\right).$$

There is an obvious similarity between Theorem 3 and Definition 4. Choosing $K = X = \mathcal{B}(V, W)$ in Definition 4 gives,

$$\operatorname{cone}\left(\{|\mathcal{U}\rangle\langle\mathcal{U}| \mid U \in \mathcal{B}(V, W)\}\right), \tag{2}$$

but the two transformations $|\mathcal{U}\rangle\langle\mathcal{U}|$ and $\mathcal{U} \odot \mathcal{U}$ are *not* the same. The two completely positive cones in Equations (1) and (2) are, however, isomorphic. This will answer Question 2.

Theorem 4. *The Choi and Sturm-Zhang completely positive cones are isomorphic in $\mathcal{B}(V, W)$ if we take $K = X = \mathcal{B}(V, W)$ in Definition 4:*

$$\operatorname{cone}\left(\{\mathcal{U} \odot \mathcal{U} \mid \mathcal{U} \in \mathcal{B}(V, W)\}\right) \cong \operatorname{cone}\left(\{|\mathcal{U}\rangle\langle\mathcal{U}| \mid U \in \mathcal{B}(V, W)\}\right).$$

Proof. "The" tensor product of X with itself another vector space denoted by $X \otimes X$. It is unique only up to isomorphism, and is characterized by the existence of an *elementary tensor* map $(x_1, x_2) \mapsto x_1 \otimes x_2$ that is "bilinear and nothing more." If $\{x_1, x_2, \ldots, x_K\}$ is a basis for X, then one way to define the tensor product of X with itself is as

$$X \otimes X \coloneqq \operatorname{span}\left(\{x_i \otimes x_j \mid 1 \leq i, j \leq K\}\right),$$

where the $x_i \otimes x_j$ are simply meaningless symbols. (For other definitions, consult Roman [7]). Now, it is known that both

$$T_1 \coloneqq \operatorname{span}\left(\{\mathcal{U} \odot \mathcal{U} \mid \mathcal{U} \in \mathcal{B}(V, W)\}\right)$$

and

$$T_2 \coloneqq \operatorname{span}\left(\{|\mathcal{U}\rangle\langle\mathcal{U}| \mid U \in \mathcal{B}(V, W)\}\right)$$

are tensor products of $X \coloneqq \mathcal{B}(V, W)$ with itself corresponding to the elementary tensor maps $\otimes = \odot$ and $\otimes = |\cdot\rangle\langle\cdot|$ respectively. As a result,

$$T_1 \cong (X \otimes X) \cong T_2.$$

It follows that there is a single linear isomorphism sending $\mathcal{U} \odot \mathcal{U}$ to $|\mathcal{U}\rangle\langle\mathcal{U}|$. □

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