

On the symmetry of induced norm cones

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ABSTRACT

Several authors have studied the problem of making an asymmetric cone symmetric through a change of inner product, and one set of positive results pertains to the class of elliptic cones. We demonstrate that the class of elliptic cones is equal to the class of induced-norm cones that arise through Jordan-isomorphism with the second-order cone, thereby showing that this symmetry result was essentially known.

KEYWORDS

Euclidean Jordan algebra, circular cone, elliptic cone, ellipsoidal cone, second-order cone, Lorentz cone, symmetric cone

AMS CLASSIFICATION

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1. Introduction

Symmetric cones are a famous class of cones for which efficient interior-point optimization algorithms are known. Optimizers have begun to ask if an asymmetric cone can be made symmetric through a change of inner product. This is an attractive idea because when it works, the protagonist feels as if he has cheated fate, transmuting a mundane cone into a desirable one at no additional cost.

From one point of view, this is a solved problem, because every symmetric cone is the cone of squares in a Euclidean Jordan algebra [1]. But an optimizer generally starts with a convenient description of a cone that interests him, not with a Euclidean Jordan algebra. Essentially he wants to know if there exists a Euclidean Jordan algebra having his cone as its cone of squares. That is a harder problem, of determining if two descriptions of a set are equivalent.

This endeavor suffered an inauspicious start when one author proposed a non-bilinear inner product that would make the p -norm cones symmetric for $p \neq 2$. That result was swiftly and simultaneously debunked by both Miao, Lin, and Chen [2], and by Ito and Lourenço [3]. The misstep proved fortuitous, however, because it encouraged Ito and Lourenço [4] to characterize the automorphism group of the p -norm cones, proving that said cones are neither homogeneous nor self-dual in non-trivial cases. This is a deeper result, and in the process the authors answered an open question posed by Gowda and Trott [5].

In 2017, Alzalg [6] succeeded in showing that the *circular cone*

$$\mathcal{L}_\theta^n := \left\{ (x_1, \bar{x})^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \tan(\theta) \geq \|\bar{x}\|_2 \right\}$$

corresponding to $\theta \in (0, \pi/2)$ is symmetric under the inner product $\langle x, y \rangle_{J_\theta} := x^T J_\theta y$, where

$$J_\theta := \text{diag} \left(1, (\cot \theta)^2, \dots, (\cot \theta)^2 \right) \in \mathbb{R}^{n \times n}.$$

Alzalg derived some Jordan-algebraic properties of the resulting symmetric cone, and partnered with Pirhaji [7] to exploit them in a family of interior-point methods for ‘circular programming’ problems. The authors [8] then extended those ideas to the family of *elliptic cones*,

$$K_M^n := \left\{ (x_1, \bar{x})^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|M\bar{x}\|_2 \right\}, \quad (1)$$

where now $M \in \mathbb{R}^{(n-1) \times (n-1)}$ is any nonsingular matrix and circular cones constitute the special case $M = \cot(\theta)I$. Again the authors show that these cones can be made symmetric, deduce some properties of the associated Jordan algebra, and devise a family of interior-point methods for them. We show that the Jordan-algebraic properties of these cones were essentially known; from the right viewpoint, elliptic cones reveal themselves to be Jordan-isomorphic to the familiar second-order cone.

Lu and Chen [9] independently and directly derived the relationship between ellipsoidal, elliptic, and second-order cones, showing that all three are symmetric with respect to appropriate inner products. In the concluding remarks of a later work with Miao [10], they contemplate the possibility that a Jordan-like multiplication underlies that relationship. But for a detail or two, the Jordan-algebra isomorphism between those cones seems to have been overlooked. In practice however, being mathematically isomorphic is not the same thing as being the same thing, so these direct analyses may still be of value to optimizers and engineers. By expounding the Jordan structure of elliptic cones, we hope to explain the recently observed phenomena and to provide inspiration for practitioners.

2. Jordan algebras

Definition 1. A *Jordan algebra* (V, \circ) is an algebra V whose bilinear ‘Jordan multiplication’ operation \circ is commutative and satisfies

$$\forall x, y \in V : x \circ ((x \circ x) \circ y) = (x \circ x) \circ (x \circ y). \quad (2)$$

If the scalar field is real and if $(x \circ x) + (y \circ y) = 0$ implies that both $x = 0$ and $y = 0$, then the Jordan algebra is *formally-real*.

Jordan algebras need not be associative, but they are *power-associative*, so expressions of the form $x^3 := x \circ (x \circ x) = (x \circ x) \circ x$ are unambiguous. There is a rich general theory of Jordan algebras, but optimizers tend to work in *Euclidean Jordan algebras* where an inner product is available. For the sake of brevity, we define a Euclidean Jordan algebra to be both finite-dimensional and unital, in accord with our main reference [11].

Definition 2. A *Euclidean Jordan algebra* $(V, \circ, \langle \cdot, \cdot \rangle)$ is a finite-dimensional real Jordan algebra with a multiplicative unit element $1_V \in V$ and an inner product that satisfies

$$\forall x, y, z \in V : \langle x \circ y, z \rangle = \langle y, x \circ z \rangle. \quad (3)$$

The *degree* of an element $x \in V$ is the dimension of the subalgebra it generates, and the *rank* of a Euclidean Jordan algebra is the maximal degree of its elements.

Example 1 (Jordan spin algebra). In $V = \mathbb{R}^n$ with the usual inner product, let $x := (x_1, \bar{x})^T \in \mathbb{R} \times \mathbb{R}^{n-1}$ be written in block form and likewise for y . Then

$$x \circ y := \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x_1 y_1 + \langle \bar{x}, \bar{y} \rangle_{\mathbb{R}^{n-1}} \\ y_1 \bar{x} + x_1 \bar{y} \end{bmatrix}$$

is a commutative bilinear operation with unit element $1_V := (1, \bar{0})^T$ satisfying [Equations \(2\)](#) and [\(3\)](#). As a result, $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$ forms a Euclidean Jordan algebra known as the *Jordan spin algebra*. It has rank two [\[11\]](#) when $n \geq 2$.

Example 2. Let \mathbb{H} and \mathbb{O} represent the fields of quaternions and octonions, respectively. If $\mathcal{H}^n(\mathbb{F})$ denotes the real vector space of n -by- n Hermitian matrices whose entries come from \mathbb{F} , then $\mathcal{H}^n(\mathbb{R})$, $\mathcal{H}^n(\mathbb{C})$, $\mathcal{H}^n(\mathbb{H})$, and $\mathcal{H}^3(\mathbb{O})$ form Euclidean Jordan algebras whose Jordan multiplication and inner product are

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle_{\mathcal{H}^n(\mathbb{F})} := \Re(\text{trace}(XY)).$$

One of the most fundamental results is that every Euclidean Jordan algebra decomposes into an orthogonal direct sum of simple subalgebras. The only surprise here should be that the summands are orthogonal (Faraut and Korányi [\[11\]](#), Proposition III.4.4). And up to isomorphism, there are only five families of simple Euclidean Jordan algebras.

Theorem 1 (Classification theorem, naïve version). *Every simple Euclidean Jordan algebra is either a Jordan spin algebra from [Example 1](#), or one of the matrix algebras in [Example 2](#).*

This result is often stated more or less as above [\[12\]](#), which can be misleading. The classification is only up to isomorphism (Faraut and Korányi [\[11\]](#), Chapter V), and even then, the term ‘isomorphism’ is subversive. As it turns out, [Definition 2](#) is an ex post facto characterization.

Theorem 2 (Faraut and Korányi [\[11\]](#), Section III.1 and Proposition VIII.4.2). *A finite-dimensional real unital Jordan algebra is formally-real if and only if there exists some inner product on it that satisfies [Equation \(3\)](#).*

Historically, formally-real Jordan algebras were the objects of interest. In finite dimensions, they’re equivalent to Euclidean Jordan algebras, but the latter is an easier definition to start with. However, one consequence of that retroactive definition is that ‘isomorphism’ refers only to an invertible linear Jordan-algebra homomorphism, and not necessarily to an isometry. It could mean nothing else, because in a formally-real Jordan algebra, there may not be an inner product to preserve. In what follows, we investigate what this means for the family of Jordan spin algebras. To avoid perpetuating the confusion, we use the term *Jordan isomorphism* to indicate an invertible linear Jordan-algebra homomorphism.

3. Induced norm cones

The classification theorem leads us to wonder what Jordan algebras are Jordan-isomorphic to our [Example 1](#). Fortunately, this is known.

Proposition 1 (Faraut and Korányi [11], Corollary IV.1.5). *Any simple Euclidean Jordan algebra of rank two is Jordan-isomorphic to an algebra associated with a positive-definite bilinear form.*

To understand what this means, one must refer to example (2) at the beginning of Faraut and Korányi's Chapter III. If \mathcal{B} is a positive-definite bilinear form (which must be symmetric to ensure the commutativity of the algebra multiplication), then the associated Euclidean Jordan algebra multiplication is

$$\begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x_1 y_1 + \mathcal{B}(\bar{x}, \bar{y}) \\ y_1 \bar{x} + x_1 \bar{y} \end{bmatrix}.$$

Recalling that every symmetric positive-definite bilinear form \mathcal{B} on $\mathbb{R}^n \times \mathbb{R}^n$ is of the form $\mathcal{B} = (x, y) \mapsto \langle Bx, y \rangle$ for some symmetric positive-definite matrix $B \in \mathbb{R}^{n \times n}$, we summarize what is known.

Theorem 3. *If $B \in \mathbb{R}^{(n-1) \times (n-1)}$ is symmetric and positive-definite for $n \geq 2$, then the operation*

$$\begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} x_1 y_1 + \langle B\bar{x}, \bar{y} \rangle_{\mathbb{R}^{n-1}} \\ y_1 \bar{x} + x_1 \bar{y} \end{bmatrix}$$

on \mathbb{R}^n defines a family of simple rank-two Euclidean Jordan algebras whose inner products are positive scalar multiples of

$$(x, y) \mapsto x_1 y_1 + \langle B\bar{x}, \bar{y} \rangle_{\mathbb{R}^{n-1}} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix}, \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} \right\rangle_{\mathbb{R}^n}$$

and whose symmetric cone of squares is

$$K := \left\{ (x_1, \bar{x})^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|\bar{x}\|_B \right\}, \text{ where } \|\bar{x}\|_B := \langle B\bar{x}, \bar{x} \rangle_{\mathbb{R}^{n-1}}^{1/2}.$$

Conversely, every simple Euclidean Jordan algebra of dimension n and rank two arises via Jordan-isomorphism from an algebra of this form.

Proof. That these are simple rank-two Euclidean Jordan algebras and that all simple rank-two Euclidean Jordan algebras are of this form follows from [Proposition 1](#) and the example to which it refers. Every inner product on a simple Euclidean Jordan algebra is a positive scalar multiple of $(x, y) \mapsto \text{trace}(x \circ y)$ per Proposition III.4.1 in Faraut and Korányi, and

$$\text{trace}(x \circ y) = \text{trace} \left(\begin{bmatrix} x_1 y_1 + \langle B\bar{x}, \bar{y} \rangle_{\mathbb{R}^{n-1}} \\ y_1 \bar{x} + x_1 \bar{y} \end{bmatrix} \right) = 2(x_1 y_1 + \langle B\bar{x}, \bar{y} \rangle_{\mathbb{R}^{n-1}})$$

is known—again from example (2) following Proposition II.2.4 in Faraut and Korányi. This justifies our description of the possible inner products. Finally, the interior of the cone of squares in these Euclidean Jordan algebras is described in example (2) following Proposition III.2.2 in Faraut and Korányi, and its closure is obviously what we have claimed. \square

We see now that the cones of squares in the simple rank-two algebras are precisely the elliptic cones from Equation (1), since for a nonsingular $M \in \mathbb{R}^{(n-1) \times (n-1)}$,

$$\|Mx\|_2^2 = \langle Mx, Mx \rangle_{\mathbb{R}^{n-1}} = \langle M^T Mx, x \rangle_{\mathbb{R}^{n-1}} = \|x\|_{M^T M}^2$$

where $B := M^T M$ is symmetric and positive-definite. Conversely, any symmetric positive-definite B can be written as $B = M^T M$ for an invertible M . Indeed, this could have been deduced directly. Jordan, von Neumann, and Wigner characterize the rank-two simple Euclidean Jordan algebras as follows.

Theorem 4 (Fundamental Theorem 2, abridged [13]). *Every simple rank-two Euclidean Jordan algebra V of dimension n consists of a basis $\{1_V, s_1, \dots, s_{n-1}\}$ and Jordan multiplication satisfying*

$$1_V \circ 1_V = 1_V, \quad 1_V \circ s_i = s_i, \quad \text{and} \quad s_i \circ s_j = \delta_{ij} 1_V.$$

Without loss of generality, we assume that V is \mathbb{R}^n . In the Euclidean Jordan algebra corresponding to an elliptic cone, Alzalg and Pirhaji [8] conclude that $1_V = (1, \vec{0})^T$ is the multiplicative unit element. It therefore automatically satisfies the first two equalities in Theorem 4. If $\{e_1, \dots, e_{n-1}\}$ is the standard basis in \mathbb{R}^{n-1} , then we can extend $\{1_V\}$ to a basis $\{1_V, s_1, \dots, s_{n-1}\}$ of \mathbb{R}^n using $(s_i)_1 := 0$ and $\vec{s}_i := M^{-1}e_i$ where M is nonsingular and satisfies $B = M^T M$ in our Theorem 3. Afterwards, we can easily check that for all i, j we have

$$s_i \circ s_j = \begin{bmatrix} \langle B\vec{s}_i, \vec{s}_j \rangle_{\mathbb{R}^{n-1}} \\ 0 \end{bmatrix} = \begin{bmatrix} \langle M^T M M^{-1}e_i, M^{-1}e_j \rangle_{\mathbb{R}^{n-1}} \\ 0 \end{bmatrix} = \delta_{ij} 1_V.$$

Thus we have recovered the family of simple rank-two algebras characterized in 1934, albeit from a different perspective. Theorem 3 also emphasizes that the elliptic cones are the only family of cones Jordan-isomorphic to the second-order cones, so further generalizations beyond elliptic cones should prove difficult. To conclude, we recall a few elementary results:

- (1) If $\|x\|_B$ denotes the norm induced by the inner product $(x, y) \mapsto \langle Bx, y \rangle$, then its dual norm is $\|x\|_{B^*} = \|x\|_{B^{-1}}$.
- (2) If $K = \left\{ (x_1, \vec{x})^T \mid x_1 \geq \|\vec{x}\| \right\}$ is any norm cone, then its dual cone is the norm cone corresponding to the dual norm, $K^* = \left\{ (x_1, \vec{x})^T \mid x_1 \geq \|\vec{x}\|_* \right\}$.

From these and the fact that we are dealing with the ‘only’ simple rank-two Euclidean Jordan algebra, most of the algebraic results for elliptic [8] and circular [7] cones follow: the form of K_M^n and its dual, the inner product that makes K_M^n symmetric, the self-duality and homogeneity of K_M^n under that inner product, its associativity, the quadratic representation in the algebra, the existence of a self-concordant barrier function, and the spectral decomposition.

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