OPTIMAL RECOVERY OF DIFFERENTIABLE FUNCTIONS BY UNIVARIATE SPLINES

MICHAEL ORLITZKY[†]

ADVISOR: DR. TATYANA SOROKINA[††]

Abstract. For special classes of smooth functions, we derive a closed-form expression for the approximation error of optimal recovery by certain polynomial splines. We also devise a technique for differentiating the approximation error, and explicitely define the so-called "worst" or extremal function, i.e. the function that realizes the upper bound of the error.

1. Introduction

Polynomial splines are of the utmost importance in numerical analysis. Finite elements are used in numerical solutions of differential equations, computer-aided geometric design, and computer-aided manufacturing. And the majority of finite elements are based on polynomial splines.

Our research goal is to develop computationally-simple splines of high approximation order. We construct these splines by solving certain optimal recovery problems, uniting the fields of optimal recovery and finite element analysis. Some progress in this direction has already been achieved [1].

Optimal recovery algorithms must produce good approximations. In the univariate setting, the majority of optimal recovery problems on important classes of functions have been solved, and their solutions are univariate polynomial splines. However, most problems have multiple solutions, and usually only one solution is known explicitly. It is both theoretically and practically useful to obtain all such solutions because they possess different properties. Generally, the computational complexity will vary while the high approximation order is preserved. Knowledge of all solutions therefore allows us to choose the simplest one.

We begin with an example based on previous research [7]. We let,

$$f\left(x\right) = \begin{cases} \frac{x^2}{2}, & x \in \left[-1, 0\right], \\ \frac{x^3}{3}, & x \in \left(0, 1\right], \end{cases}$$

so that,

[†] Dept. of Mathematics, Towson University, Towson, MD 21252, USA. michael@orlitzky.com.
[††]Dept. of Mathematics, Towson University, Towson, MD 21252, USA. tsorokina@towson.edu.

$$f(-1) = \frac{1}{2};\ f(1) = \frac{1}{3};\ f'(-1) = -1;\ f'(1) = 1.$$

This function is a member of the class $W^2[-1,1]$ which we define later. The "best" spline for this class was determined by Boyanov [3] in 1973. His spline is piecewise-linear,

$$s(f;x) = \begin{cases} -\left(x+\frac{1}{2}\right), & x \in \left[-1, -\frac{1}{2}\right], \\ -\frac{1}{6}\left(x+\frac{1}{2}\right), & x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ x - \frac{2}{3}, & x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

and uses only the information $T = \{f(-1), f(1), f'(-1), f'(1)\}$. However, there is another polynomial spline that approximates f using only the information T. We call this spline a quasi-interpolant:

$$q(f;x) = \begin{cases} -\frac{1}{12}(7x+1), & x \in [-1,0], \\ \frac{1}{12}(5x-1), & x \in (0,1]. \end{cases}$$

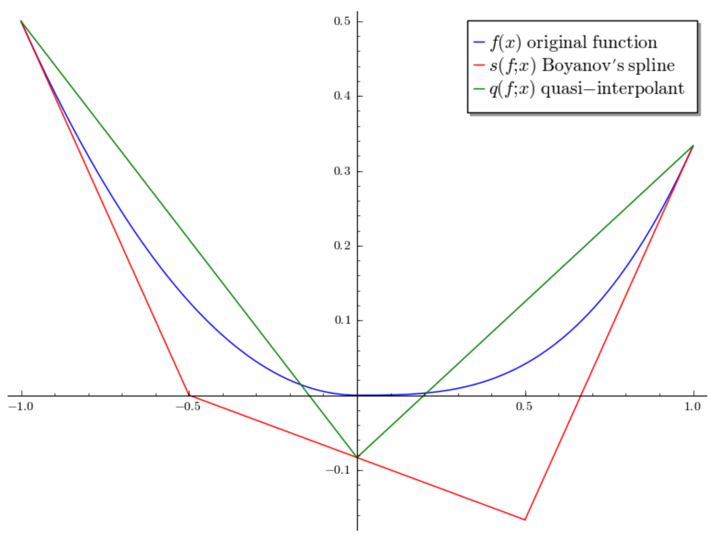

FIGURE 1.1. $f(x)$, $s(f;x)$, and $q(f;x)$.

Boyanov's spline $s(f;x)$ requires eight pieces of information to be stored, two per ordered pair,

1. $\left(-1, s(f;-1)\right)$,
2. $\left(-\frac{1}{2}, s\left(f;-\frac{1}{2}\right)\right)$,
3. $\left(\frac{1}{2}, s\left(f;\frac{1}{2}\right)\right)$,

4. $(1, s(f;1))$,

whereas the quasi-interpolant requires only six:

1. $(-1; q(f;-1))$,
2. $(0; q(f;0))$,
3. $(1, q(f;1))$.

Furthermore, q approximates f better that s does on $[0,1]$.

Each such problem has multiple piecewise-polynomial solutions. These solutions vary in smoothness, interpolation, degree, and number of knots (referred to together as *computational complexity*). We aim to find these splines by solving the associated optimal recovery problems. It is therefore necessary to be able to determine whether or not a polynomial spline is a solution to a given optimal recovery problem. We believe that this paper's results help make this determination.

This paper is organized as follows. We start with definitions and formulation of the best error bound in section 2. Then, in sections 3 and 4, we present the error bound in closed form, and show that the worst function for a class of splines is identical to the best error bound over that class. We also compute the optimal error bound for several classes of functions.

In sections 5 and 6, we present a simplified formula for the approximation error of certain splines. Derivatives of the approximation error are then computed from the simplified formula.

The first appendix contains the program code used to compute the optimal error bounds. The second appendix contains a catalog of known optimal splines.

2. Preliminaries

We will adopt the following definitions from Boyanov [3].

Definition 1. Let W^r be the space of all functions f defined over $[-1,1]$ such that $f^{(r-1)}$ is continuous, $f^{(r)}$ is piecewise continuous, and $\left\|f^{(r)}\right\|_\infty \leq 1$.

The error of a spline $s(f;x)$ at a point x is $|f(x) - s(f;x)|$, the difference between the value of the spline and the value of the function it approximates.

Definition 2. We call the following the worst error achieved by the spline s for any function in W^r,

$$e_r(s;x) = \sup_{f \in W^r} |f(x) - s(f;x)|.$$

In [3] it is shown that this supremum is attained by some function f. We call this function the "worst" function on W^r.

Definition 3. The error at x of the best spline for the worst function f in the class W^r is given by,

$$e_r^*(x) = \inf_s \left[e_r(s;x) \right].$$

The spline $S(f;x)$, found in [3], is the best spline on $[-1,1]$ using only the $0,1,\dots(r-1)^{\text{th}}$ derivatives of f at the endpoints. That is, the error of $S(f;x) = e_r^*(x)$ for all $x \in [-1,1]$. Thus, $S(f;x)$ is pointwise-optimal on the interval $[-1,1]$. Boyanov gives the worst-case error $e_r^*(x)$ of this spline,

$$e_r^*(x) = \int_{-1}^{x} (x-t)^{r-1} \operatorname{sign}\left[U_r(t)\right] \, dt,$$

where $U_r(t)$ is the polynomial of the form $t^r + a_1 t^{r-1} + \cdots + a_r$ that differs from zero least in the interval $[-1,1]$ with respect to the L_1 norm.

Our goal is to present this error in closed form. For this we will need another definition.

Definition 4. The function recursively defined by,

$$\begin{aligned} T_0(t) &= 1, \\ T_1(t) &= t, \\ &\vdots \\ T_n(t) &= 2tT_{n-1}(t) - T_{n-2}(t), \end{aligned}$$

is called the n^{th} Chebychev polynomial (of the first kind). It can be shown that this definition is equivalent to

$$T_n(t) = \cos\left(n \cdot \arccos(t)\right),$$

for $n \geq 0$.

3. Determining the sign of $U_r(t)$ on $[-1,1]$

From Powell [4], p. 175 we know that the polynomial of the form $x^{n+1}+a_1x^n+\ldots a_m$ differing least from zero in the interval $[-1,1]$ is,

$$T'_{n+2}(t) \,/\, \left[2^{n+1}(n+2)\right],$$

where $T_{n+2}(t)$ is the $(n+2)^{\text{nd}}$ Chebychev polynomial. It is clear from the two polynomials that $n+1=r$, or $n=r-1$, so we can translate this expression,

$$U_r(t) = T'_{r+1}(t) \,/\, [2^r(r+1)].$$

Now, the Chebychev polynomials are given by

$$T_{r+1}(t) = \cos[(r+1)\cdot\arccos(t)],$$

and thus,

$$T'_{r+1}(t) = (r+1)\cdot\sin[(r+1)\cdot\arccos(t)] \,/\, \sqrt{1-t^2},$$

is the polynomial we seek.

In order to compute $\operatorname{sign}(U_r(t)) = \operatorname{sign}\left(T'_{r+1}(t)\right)$, we need to know where this function is zero. First we note that $\sqrt{1-t^2}$ is always positive on $(-1,0]$, and that $r+1$ itself is positive. Therefore, we need to solve the following equation,

$$\begin{aligned} \sin[(r+1)\cdot\arccos(t)] &= 0, \\ (r+1)\cdot\arccos(t) &= k\pi,\ k=0,1\ldots \end{aligned}$$

which is equivalent to,

$$t = \cos\left(\frac{k\pi}{r+1}\right),\ k=0\ldots r+1.$$

We would like the smallest root to correspond to $k=0$, and the largest root to correspond to $k=r+1$, so we reindex the roots by replacing k with $r+1-k$,

$$t = \cos\left(\frac{(r+1-k)\pi}{r+1}\right),\ k=0\ldots r+1.$$

Finally, we denote the kth root,

$$\xi_k = \cos\left(\frac{(r+1-k)\,\pi}{r+1}\right),\ k = 0\ldots r+1.$$

We can see from this expression that -1 and 1 are roots for any $r \geq 0$, corresponding to $k = 0$ and $k = r+1$, respectively.

The next thing we would like to know is whether $U_r(t)$ is positive or negative on the first interval, $(\xi_0, \xi_1) = (-1, \xi_1)$., Since $U_r(-1) = 0$ and sign $[U_r(t)]$ changes at every root, we just need the sign of $U_r(-1+\epsilon)$, where $\epsilon > 0$ and $-1+\epsilon$ lies in this first interval. We can substitute,

$$T'_{r+1}(-1+\epsilon) = (r+1)\cdot\sin\left[(r+1)\cdot\arccos(-1+\epsilon)\right]/\sqrt{1-(-1+\epsilon)^2}.$$

The denominator is clearly positive, so,

$$\operatorname{sign}\left[T'_{r+1}(-1+\epsilon)\right] = \operatorname{sign}\left[\sin\left[(r+1)\cdot\arccos(-1+\epsilon)\right]\right].$$

$\arccos(-1)$ is π, so $\arccos(-1+\epsilon)$ is a little bit less than π. Say, $\arccos(-1+\epsilon) = \pi - \delta$. Then, since $\sin(\pi-\delta)$ is positive, $\sin[2(\pi-\delta)]$ is negative. Likewise, $\sin[3(\pi-\delta)]$ is positive. Thus,

$$\operatorname{sign}\left[T'_{r+1}(-1+\epsilon)\right] = \operatorname{sign}\left[(r+1)\cdot\sin\left[(r+1)\cdot\arccos(-1+\epsilon)\right]\right] = \operatorname{sign}\left[(-1)^{r+2}\right].$$

This gives us,

$$\operatorname{sign}\left[U_r(t)\right] = \operatorname{sign}\left[(-1)^r\right] = (-1)^r,\ t \in [\xi_0, \xi_1],$$

over the first interval of integration, and therefore, over all subintervals of $[-1, 1]$.

We consider some examples.

Example 5. We start with an example for $r = 1$.

We'd like to find the zeros of,

$$U_r(t) = T'_{r+1}(t)/\left[2^r(r+1)\right],$$

which by earlier calculations are,

$$\xi_k = \cos\left(\frac{k\pi}{2}\right) = \begin{cases} -1, & k = 0, \\ 0, & k = 1, \\ 1, & k = 2. \end{cases}$$

Assuming $-1 \leq x \leq 0$, the integral that we would like to calculate is,

$$\int_{-1}^{x} (x-t)^0 \operatorname{sign}\left[U_r(t)\right] dt.$$

Substituting our formula for $\operatorname{sign}\left[U_r(t)\right]$,

$$e_1^*(x) = \int_{-1}^{x} (-1)^r \; dt = \int_{-1}^{x} -1 \, dt = -\left[t\right]_{-1}^{x} = -\left[x+1\right] = -x-1,$$

which is the correct result.

For $0 \le x \le 1$, we have to add 1 to the exponent of (-1), since the sign will have changed at ξ_1:

$$\begin{aligned} e_1^*(x) &= \int_{-1}^{0} (-1)^r \; dt + \int_{0}^{x} (-1)^{r+1} \; dt \\ &= -1 + x. \end{aligned}$$

Thus,

$$e_1^*(x) = \begin{cases} -x-1, & -1 \le x \le 0, \\ -1+x, & 0 \le x \le 1. \end{cases}$$

4. Computing $e_r^*(x)$ and the Extremal Function on W^r

The general error function will be piecewise in x, since it depends on which interval $[\xi_i, \xi_{i+1}]$ we evaluate the function. We will assume that $x \in [\xi_i, \xi_{i+1}]$ for now. When $i = 0$, we simply ignore the term in brackets.

$$e_r^*(x) = \left[\sum_{k=0}^{i-1} (-1)^{r+k} \int_{\xi_k}^{\xi_{k+1}} (x-t)^{r-1} \; dt \right] + (-1)^{r+i} \int_{\xi_i}^{x} (x-t)^{r-1} \; dt.$$

We make the substitution $(x-t)^r = G(t)$ with $G'(t) = g(t) = r! \cdot (x-t)^{r-1} \cdot (-1)$:

$$\begin{aligned} e_r^*(x) &= \frac{1}{r!}\left\{\left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\int_{\xi_k}^{\xi_{k+1}} g(t)\ dt\right]+(-1)^{r+i-1}\int_{\xi_i}^{x} g(t)\ dt\right\} \\ &= \frac{1}{r!}\left\{\left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\left[G(t)\right]_{\xi_k}^{\xi_{k+1}}\right]+(-1)^{r+i-1}\left[G(t)\right]_{\xi_i}^{x}\right\} \\ &= \frac{1}{r!}\left\{\left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\left[(x-\xi_{k+1})^r-(x-\xi_k)^r\right]\right]+(-1)^{r+i-1}\left[(x-x)^r-(x-\xi_i)^r\right]\right\} \\ &= \frac{1}{r!}\left\{\left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\left[(x-\xi_{k+1})^r-(x-\xi_k)^r\right]\right]+(-1)^i(\xi_i-x)^r\right\}. \end{aligned}$$

Theorem 6. *The error of Boyanov's spline $S(f;x)$ on $[-1,1]$ is given piecewise by,*

$$r!\cdot e_r^*(x)=\begin{cases}(-1-x)^r, & x\in[-1,\xi_1], \\ \left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\left[(x-\xi_{k+1})^r-(x-\xi_k)^r\right]\right]+(-1)^i(\xi_i-x)^r, & x\in[\xi_i,\xi_{i+1}].\end{cases}$$

Note that by Boyanov's formula, this is negative for odd r. We can either take the absolute value or multiply everything by $(-1)^r$ to make the error non-negative.

Example 7. $r=1$. We already know the result for the W^1, so we can check. First, let $x\in[\xi_0,\xi_1]$:

$$e_r^*(x)=(\xi_0-x)^1=-1-x,$$

and for $x\in(\xi_1,\xi_2)$,

$$\begin{aligned} e_r^*(x) &= \left[(x-\xi_1)-(x-\xi_0)\right]+(-1)(\xi_1-x) \\ &= -1+x, \end{aligned}$$

as expected.

We have graphed $e_r^*(x)$ for selected values of r. The graphs have all been made non-negative.

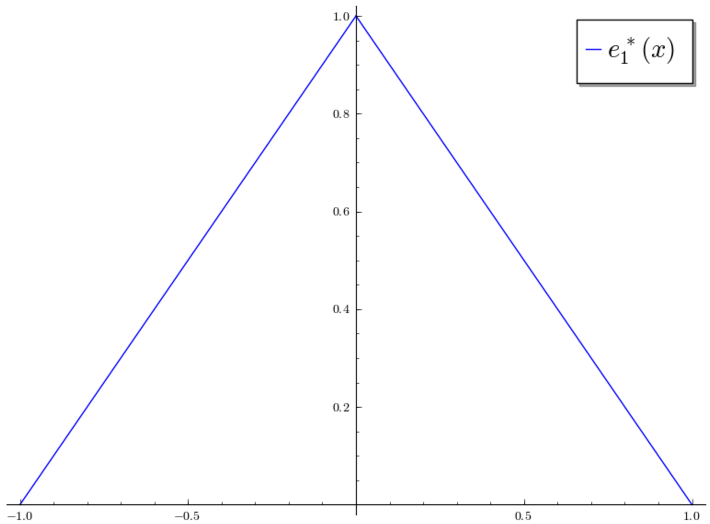

FIGURE 4.1. $e_1^*(x)$.

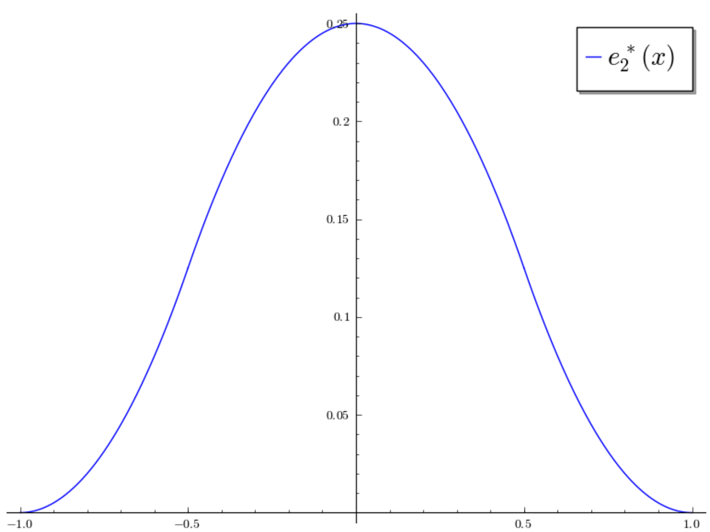

FIGURE 4.2. $e_2^*(x)$.

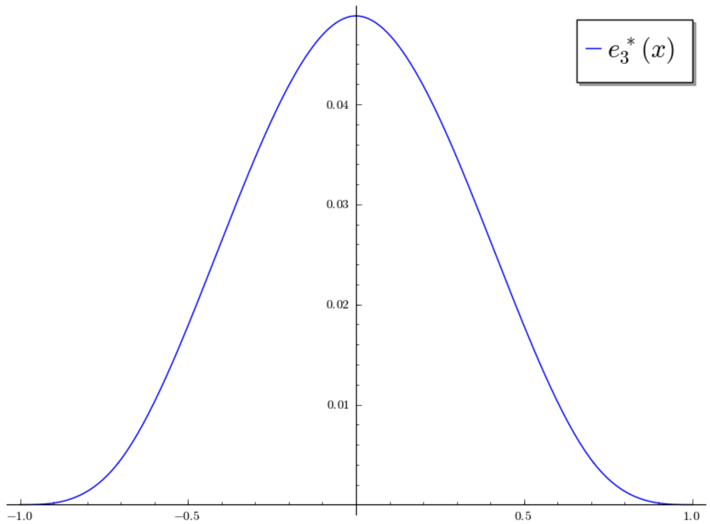

FIGURE 4.3. $e_3^*(x)$.

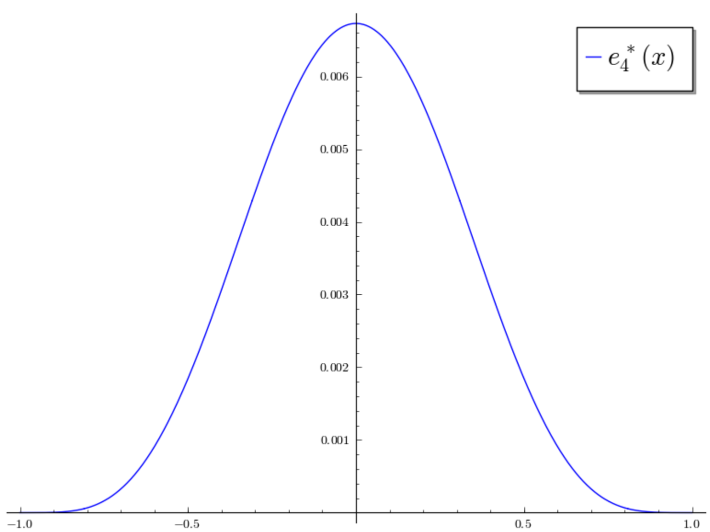

FIGURE 4.4. $e_4^*(x)$.

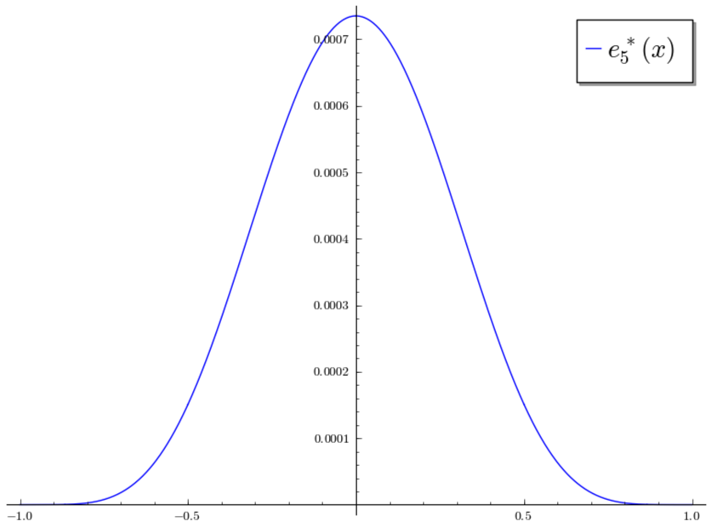

FIGURE 4.5. $e_5^*(x)$.

Theorem 8. *The function f that maximizes* $|f(x) - S(f;x)|$ is $e_r^*(x)$.

Proof. For all $x \in [-1, 1]$, the maximum possible error $|f(x) - S(f;x)|$ at x is by definition $e_r^*(x)$. Now, from Boyanov [3] we have,

$$S(f;x) = \sum_{k=0}^{r-1} A_k(x) f^{(k)}(-1) + B_k(x) f^{(k)}(1).$$

If we let $f = e_r^*$, we notice that,

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0 \ldots r-1.$$

So,

$$S(e_r^*;x) = \sum_{k=0}^{r-1} A_k(x) \cdot 0 + B_k(x) \cdot 0 \equiv 0,$$

and,

$$\begin{aligned} |f(x) - S(f;x)| &= |e_r^*(x) - S(e_r^*;x)| \\ &= |e_r^*(x)| \\ &= e_r^*(x). \end{aligned}$$

Since supremums are unique, $e_r^*(x)$ is the function maximizing $|f(x) - S(f;x)|$.

We can also show that $e_r^*(x)$ is in fact in the class W^r. Let us take the jth derivative of $e_r^*(x)$. Over the first interval $[-1,\xi_1]$ we have,

$$\frac{d^j}{dx^j}\{e_r^*(x)\} = \frac{1}{(r-j)!}(-1-x)^{r-j}(-1)^j$$

and over subsequent intervals,

$$\begin{aligned}\frac{d^j}{dx^j}\{e_r^*(x)\} &= \sum_{k=0}^{i-1}(-1)^{r+k-1}\left[\frac{1}{(r-j)!}(x-\xi_{k+1})^{r-j} - \frac{1}{(r-j)!}(x-\xi_k)^{r-j}\right] \\ &+ (-1)^{i+j}\frac{1}{(r-j)!}(\xi_i - x)^{r-j}\end{aligned}$$

We will write $\frac{1}{(r-j)!}$ as α to simplify the notation for the rest of the proof.

At $x = \xi_1$,

$$\begin{aligned}\alpha(-1-\xi_1)^{r-j}(-1)^j &= (-1)^{r-1}\left[\alpha(x-\xi_1)^{r-j} - \alpha(x-\xi_0)^{r-j}\right] \\ &+ (-1)^j\alpha(\xi_1 - x)^{r-j} \\ &= (-1)^{r-1}\left[-\alpha(\xi_1+1)^{r-j}\right] \\ &= (-1)^j\alpha(-\xi_1-1)^{r-j}\end{aligned}$$

This shows that the first and second pieces join continuously. For the rest of the pieces, we'll consider the node ξ_q with $[\xi_{q-1},\xi_q]$ on the left and $[\xi_q,\xi_{q+1}]$ on the right. On the left,

$$\begin{aligned}\frac{d^j}{dx^j}\{e_r^*(x)\} &= \sum_{k=0}^{q-2}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j} - \alpha(\xi_q-\xi_k)^{r-j}\right] \\ &+ (-1)^{q-1+j}\alpha(\xi_{q-1}-\xi_q)^{r-j}\end{aligned}$$

and on the right,

$$\begin{aligned}
\frac{d^j}{dx^j}\{e_r^*(x)\} &= \sum_{k=0}^{q-1}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j}-\alpha(\xi_q-\xi_k)^{r-j}\right]\\
&+ +(-1)^{i+j}\alpha(\xi_q-\xi_q)^{r-j}\\
&= \sum_{k=0}^{q-1}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j}-\alpha(\xi_q-\xi_k)^{r-j}\right]\\
&= \sum_{k=0}^{q-2}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j}-\alpha(\xi_q-\xi_k)^{r-j}\right]\\
&+ (-1)^{r+q-2}\left[\alpha(\xi_q-\xi_q)^{r-j}-\alpha(\xi_q-\xi_{q-1})^{r-j}\right]\\
&= \sum_{k=0}^{q-2}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j}-\alpha(\xi_q-\xi_k)^{r-j}\right]\\
&+ (-1)^{r+q-1}\alpha(\xi_q-\xi_{q-1})^{r-j}\\
&= \sum_{k=0}^{q-2}(-1)^{r+k-1}\left[\alpha(\xi_q-\xi_{k+1})^{r-j}-\alpha(\xi_q-\xi_k)^{r-j}\right]\\
&+ (-1)^{q-1+j}\alpha(\xi_{q-1}-\xi_q)^{r-j}
\end{aligned}$$

So, the rest of the pieces join continuously. Moreover, for $j = 0 \ldots r-2$, we can let $j = j+1$ to see that $\frac{d^j}{dx^j}\{e_r^*(x)\}$ is differentiable.

When $j = r-1$, we see that $\frac{d^j}{dx^j}\{e_r^*(x)\}$ is piecewise linear. Over the first interval, we have,

$$\frac{d^j}{dx^j}\{e_r^*(x)\} = (-1-x)(-1)^{r-1}$$

and over subsequent intervals,

$$\frac{d^j}{dx^j}\{e_r^*(x)\} = \frac{1}{r}\left\{\left[\sum_{k=0}^{i-1}(-1)^{r+k-1}\left[(x-\xi_{k+1})-(x-\xi_k)\right]\right]+(-1)^{i+r-1}(\xi_i-x)\right\}$$

Finally, when $j = r$,

$$\frac{d^r}{dx^r}\{e_r^*(x)\} = \begin{cases}(-1)^r, & x\in[-1,\xi_1],\\ (-1)^{i+r}, & x\in[\xi_i,\xi_{i+1}]\end{cases} = (-1)^{i+r},\ x\in[\xi_i,\xi_{i+1}]$$

□

Theorem 9. $\max\limits_{x\in[-1,1]} e_r^*(x) = e_r^*(0)$.

Proof. This follows from the proof of Boyanov's[3] Lemma 3. □

Definition 10. We say that the spline $\sigma(f;x)$ is optimal if, for all $x \in [-1,1]$,

$$e_r(\sigma;x) \leq \inf_s \sup_{f \in W^r} |f(x) - s(f;x)|.$$

Corollary 11. *A method of approximation $\sigma(f;x)$ is optimal if $\sigma(f;x) \leq e_r^*(0)$ for all $x \in [-1,1]$.*

We now calculate $e_r^*(0)$ for some values of r. We already have a general formula, so to calculate $e_r^*(0)$, we simply need to compute the interval in which zero lies. Some examples with small r should be convincing.

TABLE 1. values of i such that $0 \in [\xi_i, \xi_{i+1}]$ for small r.

r	$\xi_k,\ k = 0 \dots r-1$	$i : 0 \in [\xi_i, \xi_{i+1}]$
1	$\{-1, 0, 1\}$	$0 = \frac{r-1}{2}$
2	$\left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$	$1 = \frac{r}{2}$
3	$\left\{-1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1\right\}$	$1 = \frac{r-1}{2}$
4	$\left\{-1, -\frac{\sqrt{5}+1}{4}, -\frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}+1}{4}, 1\right\}$	$2 = \frac{r}{2}$

We conclude that, for odd r, zero lies on $\left[\xi_{\frac{r-1}{2}}, \xi_{\frac{r+1}{2}}\right]$, but for even r, it lies on $\left[\xi_{\frac{r}{2}}, \xi_{\frac{r+2}{2}}\right]$. We can unify the two cases with a definition.

Definition 12. We define a function g of r that allows us to represent these two intervals with one expression,

$$g(r) = \left[1 + (-1)^{r+1}\right]/2 = \begin{cases} 0, & r \text{ even}, \\ 1, & r \text{ odd}. \end{cases}$$

Now,

$$0 \in \left[\xi_{\frac{r-g(r)}{2}}, \xi_{\frac{r-g(r)+2}{2}}\right],\ r \geq 0.$$

If we substitute this into our general formula, we find,

$$r! \cdot e_r^*(0) = \begin{cases} -1, & r = 1, \\ \left[\sum_{k=0}^{\frac{r-g(r)}{2}-1} (-1)^{k+1} \left[(\xi_{k+1})^r - (\xi_k)^r\right]\right] + (-1)^{\frac{r-g(r)}{2}} \left(\xi_{\frac{r-g(r)}{2}}\right)^r, & r > 1. \end{cases}$$

Again we note that this is negative for odd r. To make the result positive for all r, we would multiply by $(-1)^r$.

We give algebraic results for $r = 1, \ldots 5$, and numerical results for $r = 6, \ldots 10$.

TABLE 2. $e_r^*(0)$ for certain values of r.

r	$e_r^*(0)$
1	1
2	$\frac{1}{4}$
3	$\frac{2-\sqrt{2}}{12}$
4	$\frac{3\sqrt{5}+8}{192}$
5	$\frac{17-9\sqrt{3}}{1920}$
6	$6.594836546 \cdot 10^{-5}$
7	$5.025166454 \cdot 10^{-6}$
8	$3.325387768 \cdot 10^{-7}$
9	$1.944684159 \cdot 10^{-8}$
10	$1.018642370 \cdot 10^{-9}$

5. Some Results on the Error of Approximation

Let $f \in W^r$ and let $s(f;x)$ be the spline method defined by,

$$s(f;x) = \sum_{k=0}^{r-1} A_k(x) \cdot f^{(k)}(-1) + B_k(x) \cdot f^{(k)}(1).$$

Furthermore, assume that $s(f;x)$ reproduces polynomials of degree μ. For our purposes, we will have $\mu = r$, however, we will retain both names to separate the two different concepts.

Recall that the approximation error of the spline s on the function f at a point x is given by,

$$e(f;x) = f(x) - s(f;x).$$

The barycentric coordinates of x with respect to -1 and 1 respectively are,

$$\begin{aligned} b_0(x) &= \frac{1-x}{2} \\ b_1(x) &= \frac{x+1}{2} \end{aligned}$$

It follows from this definition that $b_0(x) + b_1(x) = 1$ for all x.

In this section, we intend to use the fact that $s(f;x)$ reproduces polynomials of degree μ to derive a simplified representation of the error $e(f;x)$.

The following computation is a modification of the one given by Velikin [6]; it is modified to increase the degree of the integrand. Let the function $f \in W^r$. The Taylor expansion of f about $x = -1$ multiplied by $b_0(x)$ is,

$$f_a(x) = b_0(x) \cdot \left\{ \sum_{k=0}^{\mu-1} \frac{f^{(k)}(-1)}{k!} (x+1)^k + \frac{1}{(\mu-1)!} \int_{-1}^{x} f^{(\mu)}(t) (x-t)^{\mu-1} \, dt \right\},$$

and about $x = 1$, multiplied by $b_1(x)$ it is,

$$f_b(x) = b_1(x) \cdot \left\{ \sum_{k=0}^{\mu-1} \frac{f^{(k)}(1)}{k!} (x-1)^k - \frac{1}{(\mu-1)!} \int_{x}^{1} f^{(\mu)}(t) (x-t)^{\mu-1} \, dt \right\}.$$

If we add the two together, we find,

$$\begin{aligned} f(x) &= f_a(x) + f_b(x) \\ &= \sum_{k=0}^{\mu-1} \frac{1}{k!} \left[b_0(x) f^{(k)}(-1) (x+1)^k + b_1(x) f^{(k)}(1) (x-1)^k \right] \\ &+ \frac{1}{(\mu-1)!} \left[b_0(x) \int_{-1}^{x} f^{(\mu)}(t) (x-t)^{\mu-1} \, dt + b_1(x) \int_{x}^{1} f^{(\mu)}(t) \cdot (-1) \cdot (x-t)^{\mu-1} \, dt \right]. \end{aligned}$$

Definition 13. We define,

$$E_p(t,x) = \begin{cases} 0, & t \notin [-1,1], \\ b_0(x) \frac{(x-t)^{p-1}}{(p-1)!}, & t \in [-1,x], \\ -b_1(x) \frac{(x-t)^{p-1}}{(p-1)!}, & t \in [x,1]. \end{cases}$$

so we can rewrite f as,

$$\begin{aligned} f(x) &= \sum_{k=0}^{\mu-1} \frac{1}{k!} \left[b_0(x) f^{(k)}(-1)(x+1)^k + b_1(x) f^{(k)}(1)(x-1)^k \right] \\ &+ \int_{-1}^{1} f^{(\mu)}(t) \cdot E_\mu(t,x)\, dt. \end{aligned}$$

We would like to use the fact that $s(f;x)$ reproduces polynomials of degree μ. Since $b_0(x)$ and $b_1(x)$ are degree one,

$$\sum_{k=0}^{\mu-1} \frac{1}{k!} \left[b_0(x) f^{(k)}(-1)(x+1)^k + b_1(x) f^{(k)}(1)(x-1)^k \right]$$

is a degree μ polynomial in x. So, $s(f;x)$ must reproduce that part of f. Therefore, by (1.2),

$$\begin{aligned} e(f;x) &= \int_{-1}^{1} f^{(\mu)}(t) \cdot E_\mu(t,x)\, dt \\ &- \sum_{i=0}^{r-1} A_i \frac{d^i}{dx^i} \left\{ \int_{-1}^{1} f^{(\mu)}(t) \cdot E_\mu(t,-1)\, dt \right\} + B_i \frac{d^i}{dx^i} \left\{ \int_{-1}^{1} f^{(\mu)}(t) \cdot E_\mu(t,1) \right\} dt. \end{aligned}$$

Moving the integration outside of the sum,

$$\begin{aligned} e(f;x) &= \int_{-1}^{1} f^{(\mu)}(t) \cdot E_\mu(t,x)\, dt \\ &- f^{(\mu)}(t) \left[\sum_{i=0}^{r-1} A_i \frac{d^i}{dx^i} \{E_\mu(t,-1)\} + B_i \frac{d^i}{dx^i} \{E_\mu(t,1)\}\, dt \right]. \end{aligned}$$

We recognize the term in square brackets,

$$\begin{aligned} e(f;x) &= \int_{-1}^{1} f^{(\mu)}(t) \{E_\mu(t,x) - s[E_\mu(t,x)]\}\, dt \\ &= \int_{-1}^{1} f^{(\mu)}(t)\, e[E_\mu(t,x);x]\, dt. \end{aligned}$$

Definition 14. We set,

$$Q_p(t,x) = e[E_p(t,x);x]$$

to be the approximation error of the function E_p by the spline s. If we assume that $t \in [-1, x]$, then from the definition of e,

$$Q_p(t,x) = b_0(x)\frac{(x-t)^{p-1}}{(p-1)!} - s\left[E_p(t,x);x\right].$$

Before we go any further, we would like to know what the derivatives with repsect to x of E_p look like. Assuming $-1 \le t \le x$,

$$\begin{aligned}
\frac{d}{dx}E_p(t,x) &= b_0'(x)\frac{(x-t)^{p-1}}{(p-1)!} + b_0(x)\frac{(x-t)^{p-1-1}}{(p-1-1)!}\\
&= -\frac{(x-t)^{p-1}}{2(p-1)!} + E_{p-1}(t,x);\\
\frac{d^2x}{dx^2}E_p(t,x) &= \frac{d}{dx}\left\{-\frac{(x-t)^{p-1}}{2(p-1)!} + E_{p-1}(t,x)\right\}\\
&= \frac{d}{dx}\left\{-\frac{(x-t)^{p-1}}{2(p-1)!}\right\} + \frac{d}{dx}\left\{E_{p-1}(t,x)\right\}\\
&= -\frac{(x-t)^{p-1-1}}{2(p-1-1)!} + \frac{d}{dx}\left\{E_{p-1}(t,x)\right\}\\
&= -\frac{(x-t)^{p-1-1}}{2(p-1-1)!} + \left[-\frac{(x-t)^{p-1-1}}{2(p-1-1)!} + E_{p-1-1}(t,x)\right]\\
&= -\frac{(x-t)^{p-1-1}}{(p-1-1)!} + E_{p-1-1}(t,x);\\
\frac{d^3x}{dx^3}E_p(t,x) &= \frac{d}{dx}\left\{-\frac{(x-t)^{p-1-1}}{(p-1-1)!} + E_{p-1-1}(t,x)\right\}\\
&= -\frac{(x-t)^{p-1-2}}{(p-1-2)!} + \frac{d}{dx}\left\{E_{p-1-1}(t,x)\right\}\\
&= -\frac{(x-t)^{p-1-2}}{(p-1-2)!} + \left[-\frac{(x-t)^{p-1-2}}{2(p-1-2)!} + E_{p-1-2}(t,x)\right]\\
&= -\frac{3(x-t)^{p-1-2}}{2(p-1-2)!} + E_{p-1-2}(t,x);\\
&\dots\\
\frac{d^kx}{dx^k}E_p(t,x) &= -\frac{k(x-t)^{p-k}}{2(p-k)!} + E_{p-k}(t,x);\\
&\dots\\
\frac{d^px}{dx^p}E_p(t,x) &= \text{constant};\\
\frac{d^{p+1}x}{dx^{p+1}}E_p(t,x) &= 0.
\end{aligned}$$

Now we assume that $x \le t \le 1$:

$$
\begin{aligned}
\frac{d}{dx}E_p\left(t,x\right) &= -b_1'\left(x\right)\frac{\left(x-t\right)^{p-1}}{\left(p-1\right)!}-b_1\left(x\right)\frac{\left(x-t\right)^{p-1-1}}{\left(p-1-1\right)!} \\
&= -\frac{\left(x-t\right)^{p-1}}{2\left(p-1\right)!}+E_{p-1}\left(t,x\right); \\
\frac{d^2}{dx^2}E_p\left(t,x\right) &= -\frac{\left(x-t\right)^{p-1-1}}{2\left(p-1-1\right)!}+\frac{d}{dx}\left\{E_{p-1}\left(t,x\right)\right\} \\
&= -\frac{\left(x-t\right)^{p-1-1}}{2\left(p-1-1\right)!}+\left[-\frac{\left(x-t\right)^{p-1-1}}{2\left(p-1-1\right)!}+E_{p-1-1}\left(t,x\right)\right] \\
&= -\frac{2\left(x-t\right)^{p-1-1}}{2\left(p-1-1\right)!}+E_{p-1-1}\left(t,x\right); \\
&\cdots \\
\frac{d^k}{dx^k} &= -\frac{k\left(x-t\right)^{p-k}}{2\left(p-k\right)!}+E_{p-k}\left(t,x\right).
\end{aligned}
$$

Just as we had in the first case.

Remark. Note, however, that the E_{p-k} terms are different in the two cases! In general,

$$
\frac{d^k}{dx^k}E_p\left(t,x\right)=\begin{cases}-\frac{k(x-t)^{p-k}}{2(p-k)!}+b_0\left(x\right)\frac{(x-t)^{p-k-1}}{(p-k-1)!}, & t\in\left[-1,x\right], \\ -\frac{k(x-t)^{p-k}}{2(p-k)!}-b_1\left(x\right)\frac{(x-t)^{p-k-1}}{(p-k-1)!}, & t\in\left[x,1\right].\end{cases}
$$

Now, when $p\leq\mu$, the first term will be reproduced by our spline s. Therefore, we can replace it with,

$$
\begin{aligned}
b_0\left(x\right)\frac{\left(x-t\right)^{p-1}}{\left(p-1\right)!} &= s\left[b_0\left(x\right)\frac{\left(x-t\right)^{p-1}}{\left(p-1\right)!};x\right] \\
&= \sum_{k=0}^{r-1}A_k\left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!}+b_0\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!}\right] \\
&+ \sum_{k=0}^{r-1}B_k\left[-\frac{k\left(1-t\right)^{p-k}}{2\left(p-k\right)!}+b_0\left(1\right)\frac{\left(1-t\right)^{p-k-1}}{\left(p-k-1\right)!}\right].
\end{aligned}
$$

Now, this is not the same as $s\left[E_p\left(t,x\right)\right]$ since the latter changes at $t=x$. However ,we know $s\left[E_p\left(t,x\right)\right]$, too:

$$
s\left[E_p\left(t,x\right)\right] = \sum_{k=0}^{r-1}A_k\frac{d^k}{dx^k}\left\{E_p\right\}\left(t,-1\right)+B_k\frac{d^k}{dx^k}\left\{E_p\right\}\left(t,1\right).
$$

We notice that, at $x=-1$ and for all $t\in\left[-1,1\right]$, $t\geq x$. Likewise, for $x=1$ we have $t\leq x$. Thus,

$$\begin{aligned} s\left[E_p\left(t,x\right)\right] &= \sum_{k=0}^{r-1} A_k \left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!} - b_1\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &+ \sum_{k=0}^{r-1} B_k \left[-\frac{k\left(1-t\right)^{p-k}}{2\left(p-k\right)!} + b_0\left(1\right)\frac{\left(1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right]. \end{aligned}$$

Substituting,

$$\begin{aligned} Q_p\left(t,x\right) &= \sum_{k=0}^{r-1} A_k \left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!} + b_0\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &+ \sum_{k=0}^{r-1} B_k \left[-\frac{k\left(1-t\right)^{p-k}}{2\left(p-k\right)!} + b_0\left(1\right)\frac{\left(1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &- \sum_{k=0}^{r-1} A_k \left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!} - b_1\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &- \sum_{k=0}^{r-1} B_k \left[-\frac{k\left(1-t\right)^{p-k}}{2\left(p-k\right)!} + b_0\left(1\right)\frac{\left(1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right]. \end{aligned}$$

The B_k terms cancel, and we can combine the A_k terms to get,

$$\begin{aligned} Q_p\left(t,x\right) &= \sum_{k=0}^{r-1} A_k \left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!} + b_0\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &- \sum_{k=0}^{r-1} A_k \left[-\frac{k\left(-1-t\right)^{p-k}}{2\left(p-k\right)!} - b_1\left(-1\right)\frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!} \right] \\ &= \sum_{k=0}^{r-1} A_k \frac{\left(-1-t\right)^{p-k-1}}{\left(p-k-1\right)!}. \end{aligned}$$

Now, for $t \in [x, 1]$,

$$Q_p\left(t,x\right) = -b_1\left(x\right)\frac{\left(x-t\right)^{p-1}}{\left(p-1\right)!} - s\left[E_p\left(t,x\right)\right].$$

And by similar reasoning,

$$
\begin{aligned}
-b_1(x)\frac{(x-t)^{p-1}}{(p-1)!} &= s\left[-b_1(x)\frac{(x-t)^{p-1}}{(p-1)!};x\right] \\
&= \sum_{k=0}^{r-1} A_k\left[-\frac{k(-1-t)^{p-k}}{2(p-k)!}-b_1(-1)\frac{(-1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&+ \sum_{k=0}^{r-1} B_k\left[-\frac{k(1-t)^{p-k}}{2(p-k)!}-b_1(1)\frac{(1-t)^{p-k-1}}{(p-k-1)!}\right].
\end{aligned}
$$

Since the $s[E_p(t,x)]$ term doesn't change,

$$
\begin{aligned}
Q_p(t,x) &= \sum_{k=0}^{r-1} A_k\left[-\frac{k(-1-t)^{p-k}}{2(p-k)!}-b_1(-1)\frac{(-1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&+ \sum_{k=0}^{r-1} B_k\left[-\frac{k(1-t)^{p-k}}{2(p-k)!}-b_1(1)\frac{(1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&- \sum_{k=0}^{r-1} A_k\left[-\frac{k(-1-t)^{p-k}}{2(p-k)!}-b_1(-1)\frac{(-1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&- \sum_{k=0}^{r-1} B_k\left[-\frac{k(1-t)^{p-k}}{2(p-k)!}+b_0(1)\frac{(1-t)^{p-k-1}}{(p-k-1)!}\right].
\end{aligned}
$$

Here the A_k terms will cancel and the B_k terms will combine:

$$
\begin{aligned}
Q_p(t,x) &= \sum_{k=0}^{r-1} B_k\left[-\frac{k(1-t)^{p-k}}{2(p-k)!}-b_1(1)\frac{(1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&- \sum_{k=0}^{r-1} B_k\left[-\frac{k(1-t)^{p-k}}{2(p-k)!}+b_0(1)\frac{(1-t)^{p-k-1}}{(p-k-1)!}\right] \\
&= \sum_{k=0}^{r-1} -B_k\left[b_0(1)+b_1(1)\right]\frac{(1-t)^{p-k-1}}{(p-k-1)!} \\
&= \sum_{k=0}^{r-1} -B_k\frac{(1-t)^{p-k-1}}{(p-k-1)!}.
\end{aligned}
$$

Definition 15. We give names to the two cases for $Q_p(t,x)$:

$$
\begin{aligned}
q_p(t,x) &= \sum_{k=0}^{r-1} A_k\frac{(-1-t)^{p-k-1}}{(p-k-1)!}, \\
\bar{q}_p(t,x) &= \sum_{k=0}^{r-1} -B_k\frac{(1-t)^{p-k-1}}{(p-k-1)!}.
\end{aligned}
$$

When $p-k-1<0$ the individual terms will be zero, since we will have taken the derivative of either $(1+t)^0$ or $(1-t)^0$. We note that $q_p(x,x)=\bar{q}_p(x,x)$ so we name this function $q_p(x)$.

Theorem 16. *The approximation error $e(f;x)$ of functions $f\in W^r$ by a spline s which reproduces polynomials of degree μ is bounded by,*

$$\begin{aligned} e_r(s;x) &= \int_{-1-x}^{0}\left|\sum_{k=0}^{r-1}A_{r-k-1}(x)\frac{z^{(\mu-r+k)}}{(\mu-r+k)!}\right|dz \\ &+ \int_{0}^{1-x}\left|\sum_{k=0}^{r-1}B_{r-k-1}(x)\frac{z^{(\mu-r+k)}}{(\mu-r+k)!}\right|dz. \end{aligned}$$

Proof. Recall,

$$e_r(s;x)=\sup_{f\in W^r}|f(x)-s(f;x)|=\sup_{f\in W^r}|e(f;x)|$$

If we substitute,

$$\begin{aligned} e(f;x) &= \int_{-1}^{1}f^{(\mu)}(t)\,e\left[E_\mu(t,x);x\right]\,dt \\ &= \int_{-1}^{1}f^{(\mu)}(t)\,Q_\mu(t,x)\,dt, \end{aligned}$$

we find,

$$\begin{aligned} e_r(s;x) &= \sup_{f\in W^r}\left|\int_{-1}^{1}f^{(\mu)}(t)\,Q_\mu(t,x)\,dt\right| \\ &= \int_{-1}^{1}|Q_\mu(t,x)|\,dt \\ &= \int_{-1}^{x}|q_p(t,x)|\,dt+\int_{x}^{1}|\bar{q}_p(t,x)|\,dt. \end{aligned}$$

After two changes of variable, the result follows. □

6. Computing Derivatives of the Error

Using the previous definition, we can rewrite,

$$\begin{aligned} e\left(f;x\right) &= \int_{-1}^{1} f^{(\mu)}\left(t\right) Q_{\mu}\left(t,x\right)\, dt \\ &= \int_{-1}^{x} f^{(\mu)}\left(t\right) Q_{\mu}\left(t,x\right)\, dt + \int_{x}^{1} f^{(\mu)}\left(t\right) Q_{\mu}\left(t,x\right)\, dt \\ &= \int_{-1}^{x} f^{(\mu)}\left(t\right) q_{\mu}\left(t,x\right)\, dt + \int_{x}^{1} f^{(\mu)}\left(t\right) \bar{q}_{\mu}\left(t,x\right)\, dt \\ &= \int_{-1}^{x} f^{(\mu)}\left(t\right) q_{\mu}\left(t,x\right)\, dt - \int_{1}^{x} f^{(\mu)}\left(t\right) \bar{q}_{\mu}\left(t,x\right)\, dt \\ &= I_1 - I_2 \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \int_{-1}^{x} f^{(\mu)}\left(t\right) q_{\mu}\left(t,x\right)\, dt, \\ I_2 &= \int_{1}^{x} f^{(\mu)}\left(t\right) \bar{q}_{\mu}\left(t,x\right)\, dt. \end{aligned}$$

We can use Leibniz' rule,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g\left(t,x\right)\, dt = \frac{d}{dx}\left\{b\left(x\right)\right\}\cdot g\left(b\left(x\right),x\right) - \frac{d}{dx}\left\{a\left(x\right)\right\}\cdot g\left(a\left(x\right),x\right) + \int_{a(x)}^{b(x)} \frac{d}{dx} g\left(t,x\right)\, dt$$

to calculate $e'\left(f;x\right)$. First, we calculate,

$$\begin{aligned} \frac{d}{dx} q_p\left(t,x\right) &= \sum_{k=0}^{r-1} A_k' \frac{(-1)^{p-k-1}}{(p-k-1)!}\left(1+t\right)^{p-k-1}, \\ \frac{d}{dx} \bar{q}_p\left(t,x\right) &= \sum_{k=0}^{r-1} - B_k' \frac{(1-t)^{p-k-1}}{(p-k-1)!}. \end{aligned}$$

Theorem 17. *The derivative of the approximation error is,*

$$e'(f;x) = \int_{-1}^{1} f^{(\mu)}(t) \frac{d}{dx}\{Q_\mu(t,x)\}.$$

Proof. We can apply Leibniz' rule to I_1 and I_2 independently. For I_1, we have $a(x) = -1$, $b(x) = x$, and $g = f^{(\mu)}(t)\, q_\mu(t,x)$. Thus,

$$\frac{d}{dx} I_1 = \frac{d}{dx}\left\{f^{(\mu)}(x)\, q_\mu(x,x)\right\} + \int_{-1}^{x} f^{(\mu)}(t) \frac{d}{dx}\{q_\mu(t,x)\}\, dt.$$

For I_2, $a(x) = 1$, $b(x) = x$, and $g = f^{(\mu)}(t)\, \bar{q}(t,x)$. So,

$$\frac{d}{dx} I_2 = \frac{d}{dx}\left\{f^{(\mu)}(x)\, \bar{q}_\mu(x,x)\right\} + \int_{1}^{x} f^{(\mu)}(t) \frac{d}{dx}\{\bar{q}_\mu(t,x)\}\, dt.$$

Since $q_\mu(x,x) = \bar{q}_\mu(x,x) = q_\mu(x)$,

$$\begin{aligned}
\frac{d}{dx}\{I_1 - I_2\} &= \frac{d}{dx}\left\{f^{(u)}(x)\left[q'_\mu(x) - q'_\mu(x)\right]\right\} \\
&+ \int_{-1}^{x} f^{(\mu)}(t) \frac{d}{dx}\{q_\mu(t,x)\}\, dt \\
&- \int_{1}^{x} f^{(\mu)}(t) \frac{d}{dx}\{\bar{q}_\mu(t,x)\}\, dt \\
&= \int_{-1}^{x} f^{(\mu)}(t) \frac{d}{dx}\{q_\mu(t,x)\}\, dt + \int_{x}^{1} f^{(\mu)}(t) \frac{d}{dx}\{\bar{q}_\mu(t,x)\}\, dt.
\end{aligned}$$

Since $Q_\mu(t,x)$ is defined piecewise to be $q_\mu(t,x)$ over the first interval of integration and $\bar{q}(t,x)$ over the second, we can combine the two,

$$e'(f;x) = \frac{d}{dx}\{I_1 - I_2\} = \int_{-1}^{1} f^{(\mu)}(t) \frac{d}{dx}\{Q_\mu(t,x)\}.$$

□

This can be extended to higher derivatives in a similar fashion.

References

[1] Borodachov, S. and Sorokina, T. An Optimal Multivariate Spline Method of Recovery of Twice Differentiable Functions. BIT Numerical Mathematics, volume 51, number 3, pp. 497-511. Springer, 2011.

[2] Borodachov, S. and Sorokina, T. Optimal Spline Recovery Methods for Twice Differentiable Univariate Functions. Towson University, 2011.

[3] Boyanov, B. D. Best Methods of Interpolation for Certain Classes of Differentiable Functions. Mathematical Notes, volume 17, issue 4, pp. 301-309. MAIK Nauka/Interperiodica, 1975.

[4] Powell, M. J. D. Approximation Theory and Methods. Cambridge University Press, Cambridge, 1981.

[5] Stein, William A. et al. Sage Mathematics Software (Version 5.0), The Sage Development Team, 2012, http://www.sagemath.org.

[6] Velikin, V.L. Precise Approximation Values by Hermitian Splines on Classes of Differentiable Functions. Mathematics of the USSR - Izvestiya, 7(1):163, (1973).

[7] Weaver, K. Optimal Recovery Splines for Differentiable Univariate Functions. Towson University, 2011.

Appendix 1: Sage[5] code used to compute $e_r^*(0)$.

```
"""

e_r_star_zero.py

This program computes the error of Boyanov's spline at x=0. It
can be run with the sage math suite, available from,

  http://www.sagemath.org/

It takes `r` as a command-line argument. An example:

  $ sage e_r_star_zero.py 2
  1/4

"""

from sage.all import *

if len(sys.argv) != 2:
    # Fail if `r` not given.
    print "Usage: sage e_r_star_zero.py <r>"
    sys.exit()

def derivative_roots(n):
    """
    Find the roots of the derivative of the `n`th Chebychev polynomial.
    """
    roots = []
    for k in range(0, n+1):
        root = cos(k*pi/n)
        roots.append(root)
    roots.sort()
    return roots

def g(r):
    """
    One when `r` is odd, zero when `r` is even.
    """
    return (1 + (-1)**(r+1))/QQ(2)

def e_r_star_zero(r):
    """
    Find the error of Boyanov's spline at zero.
```

```
    """

    if (r == 1):
        return 1

    xi = derivative_roots(r+1)
    result = SR(0)
    i = (r - g(r)) / 2

    for k in range(0, i):
        k = ZZ(k)
        term  = QQ(-1) ** (k+1)
        term *= ( (xi[k+1])**r - (xi[k])**r )
        result += term

    result += QQ(-1)**i * (xi[i])**r
    result *= QQ(1)/r.factorial()

    # Makes the result positive.
    result *= (-1)**r

    return result.full_simplify()

r = ZZ(sys.argv[1])
print e_r_star_zero(r)
```

Appendix 2: Catalog of Optimal Splines

We say that a function f has smoothness n if $f \in C^n$. If f is discontinuous, we say that it has smoothness -1.

W^1

In W^1, our optimal error bound is just $e_2^*(0) = 1$.

TABLE 3. List of optimal splines on W^1

Description	Degree	Knots	Interpolation	Smoothness	Score
Boyanov	0	1	v	-1	-1
Linear	1	0	v	1	1
Midpoint	0	0		0	0
Cubic	3	0	v	2	0

Boyanov. Boyanov's spline on W^1 only reproduces constants.

$$S(f;x) = \begin{cases} f(-1), & x \in [-1,0] \\ f(1), & x \in (0,1] \end{cases}$$

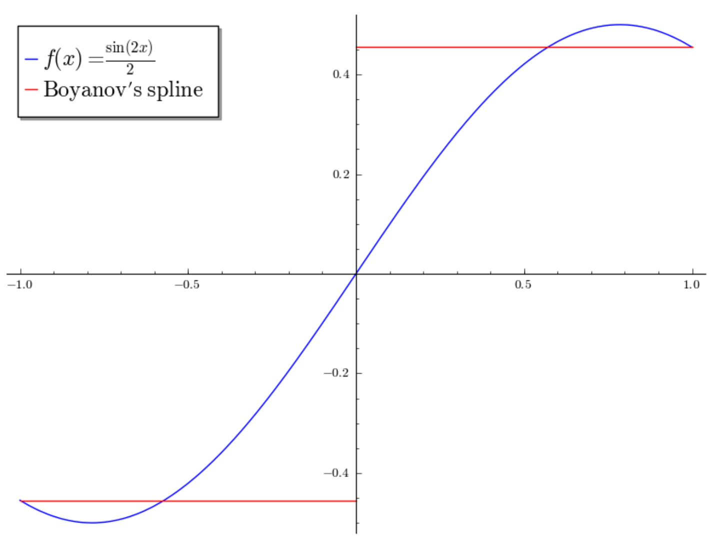

FIGURE 6.1. $S\left(\frac{\sin(2x)}{2}; x\right)$

Linear. The linear spline is just the straight line from $f(-1)$ to $f(1)$. It will therefore reproduce a linear polynomial.

$$s(f;x) = \frac{(1-x)}{2} f(-1) + \frac{(x+1)}{2} f(1)$$

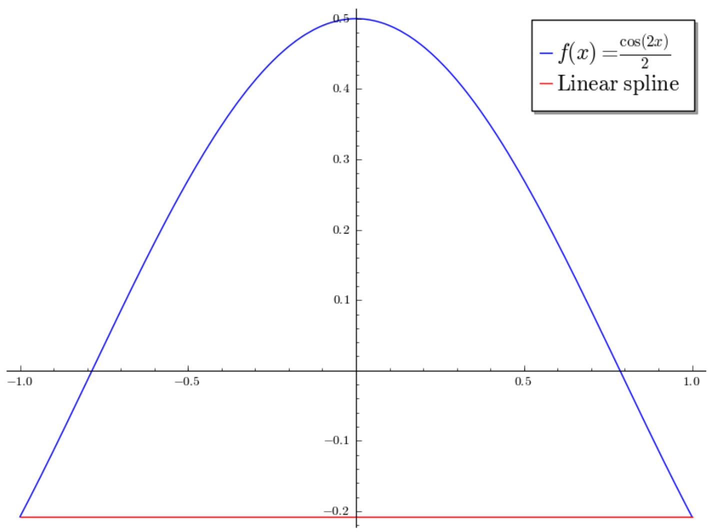

FIGURE 6.2. $s\left(\frac{\cos(2x)}{2}; x\right)$

Midpoint. The midpoint method is a horizontal line–the constant function lying midway between $f(-1)$ and $f(1)$. This is given in Boyanov's Lemma 1, an existence proof.

$$s(f;x) = \frac{f(-1) + f(1)}{2}$$

It reproduces constants.

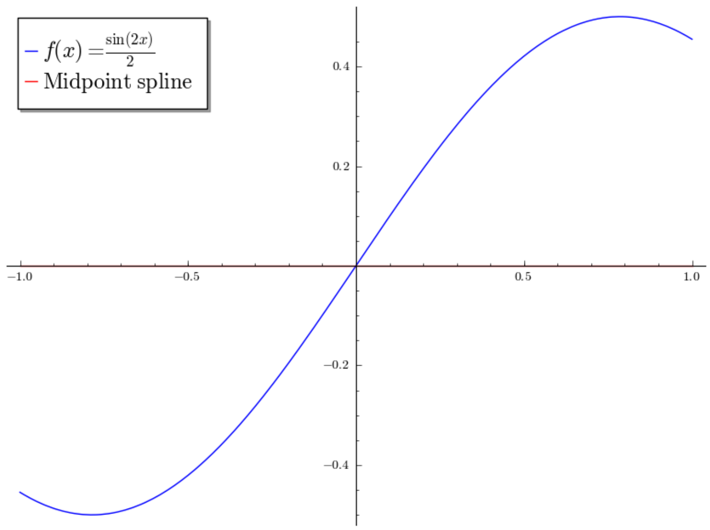

FIGURE 6.3. $s\left(\frac{\sin(2x)}{2};x\right)$

Cubic. The cubic spline on W^1 is the natural cubic through the values of Boyanov's spline at $x=-1,-\frac{1}{2},\frac{1}{2},1$. It is interesting because it disproves the conjecture that an optimal spline cannot have degree greater than $r+1$.

$$s\left(f;x\right)=\frac{1}{6}\left[\left(4x^3-7x+3\right)f\left(-1\right)-\left(4x^3-7x-3\right)f\left(1\right)\right]$$

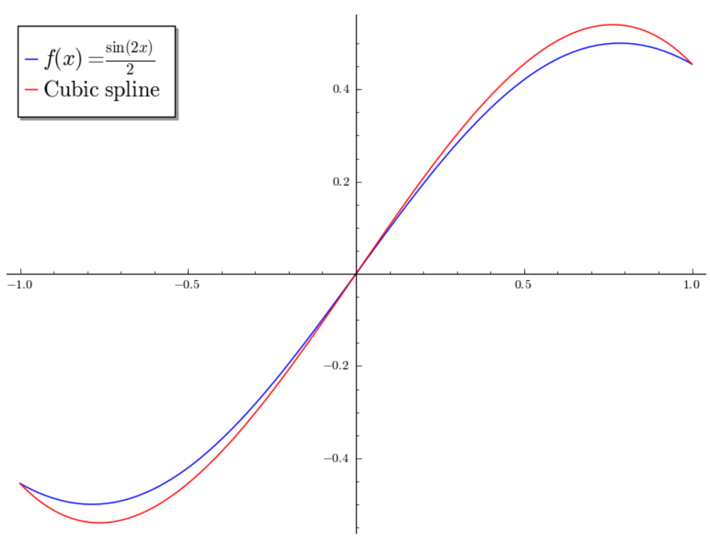

FIGURE 6.4. $s\left(\frac{\sin(2x)}{2};x\right)$

Error. The suprema of the pointwise error for each of the splines on W^1 are given below.

$$\begin{aligned} e_1(\text{linear}; x) &= \frac{1}{2}\left[(x+1)|x-1| - (x-1)|x+1|\right] \\ e_1(\text{midpoint}; x) &= 1 \\ e_1(\text{cubic}; x) &= \frac{1}{6}\left[(x+1)\left|4x^3 - 7x + 3\right| - (x-1)\left|4x^3 - 7x - 3\right|\right] \end{aligned}$$

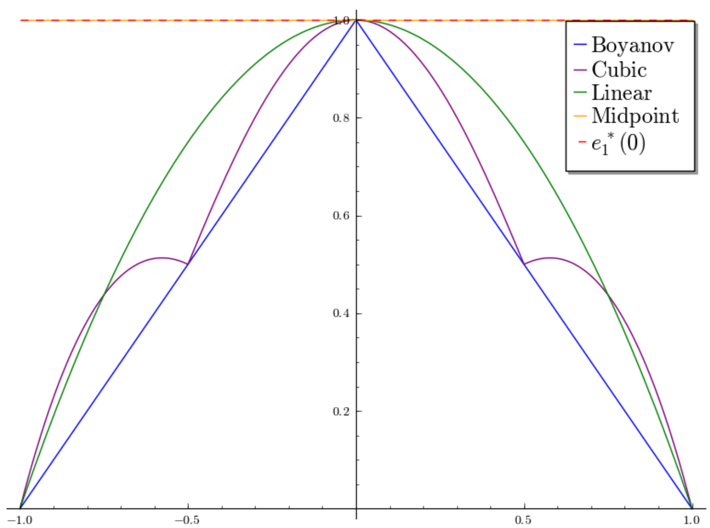

FIGURE 6.5. Error of optimal splines on W^1

W^2

For W^2, our optimal error bound is $e_2^*(0) = \frac{1}{4}$.

TABLE 4. List of optimal splines on W^2

Description	Degree	Knots	Interpolation	Smoothness	Score
Boyanov	1	2	v,d	0	-1
Quasi Quadratic	2	0	v	1	0
Cubic	3	0	v,d	2	1
Double Quadratic	2	1	v,d	1	0

Boyanov. Boyanov's spline on W^2 recovers linear polynomials.

$$S\left(f;x\right)=\begin{cases}\left(x+1\right)f'\left(-1\right)+f\left(-1\right), & x\in\left[-1,-\frac{1}{2}\right)\\ \frac{1}{2}\left[\left(1-2x\right)f\left(-1\right)+\left(1+2x\right)f\left(1\right)+\frac{(1-2x)f'(-1)-(2x+1)f'(1)}{2}\right], & x\in\left[-\frac{1}{2},\frac{1}{2}\right]\\ \left(x-1\right)f'\left(1\right)+f\left(1\right), & x\in\left(\frac{1}{2},1\right]\end{cases}$$

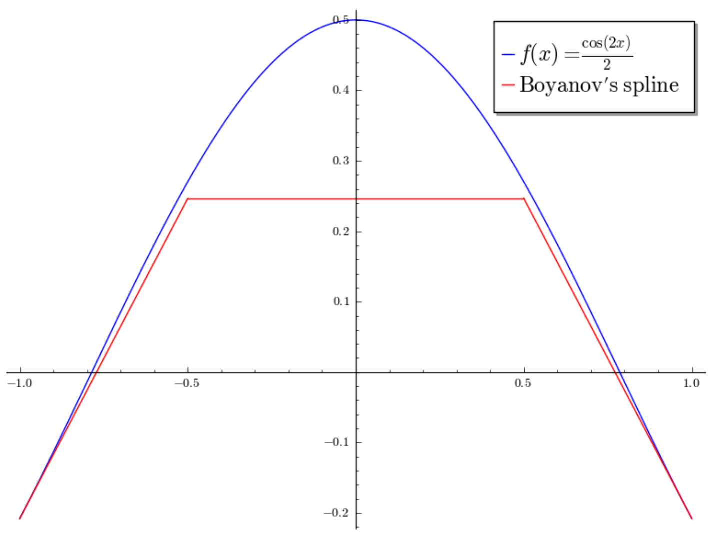

FIGURE 6.6. $S\left(\frac{\sin(2x)}{2};x\right)$

Quasi-Quadratic. This quasi-quadratic spline on W^2 interpolates the values at the endpoints, and the value of Boyanov's spline at $x=0$. It reproduces quadratic polynomials, and was described in [1].

$$s\left(f;x\right)=A_0f\left(a\right)+A_1f'\left(a\right)+B_0f\left(b\right)+B_1f'\left(b\right)$$

where,

$$\begin{aligned}A_0&=\frac{1-x}{2}\\ A_1&=\frac{1-x^2}{4}\\ B_0&=\frac{x+1}{2}\\ B_1&=\frac{x^2-1}{4}\end{aligned}$$

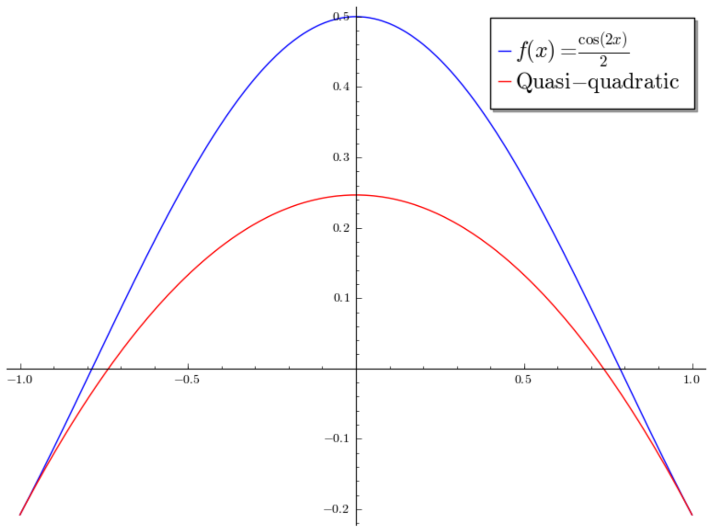

FIGURE 6.7. $s\left(\frac{\cos(2x)}{2};x\right)$

Cubic Spline. The cubic is the natural spline arising from interpolation of values and derivatives at the endpoints. We have four data, so the four coefficients correspond to those of a cubic polynomial. It naturally reproduces cubics.

$$s\left(f;x\right) = A_0 f\left(a\right) + A_1 f'\left(a\right) + B_0 f\left(b\right) + B_1 f'\left(b\right)$$

where,

$$\begin{aligned} A_0 &= \frac{x^3 - 3x + 2}{4} \\ A_1 &= \frac{x^3 - x^2 - x + 1}{4} \\ B_0 &= \frac{-x^3 + 3x + 2}{4} \\ B_1 &= \frac{x^3 + x^2 - x - 1}{4} \end{aligned}$$

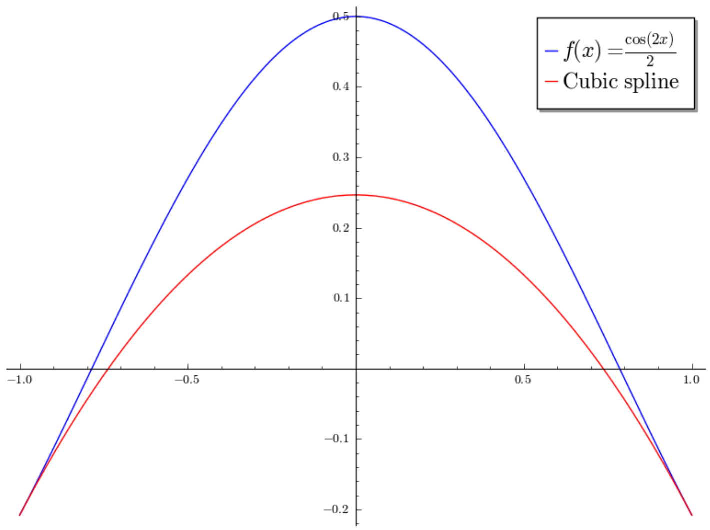

FIGURE 6.8. $s\left(\frac{\cos(2x)}{2}; x\right)$

Double Quadratic. The double quadratic is made up of two quadratic functions that join smoothly over $[-1, 0]$ and $(0, 1]$.

$$s\,(f;x) = A_0 f\,(a) + A_1 f'\,(a) + B_0 f\,(b) + B_1 f'\,(b)$$

where,

$$A_0 = \begin{cases} \frac{-x^2-x+1}{2}, & x \in [-1,0] \\ \frac{x^2-x+1}{2}, & x \in [0,1] \end{cases}$$

$$A_1 = \begin{cases} \frac{-3x^2-2x+1}{4}, & x \in [-1,0] \\ \frac{x^2+-2x+1}{4}, & x \in [0,1] \end{cases}$$

$$B_0 = \begin{cases} \frac{x^2+x+1}{2}, & x \in [-1,0] \\ \frac{-x^2+x+1}{2}, & x \in [0,1] \end{cases}$$

$$B_1 = \begin{cases} \frac{-x^2-2x-1}{4}, & x \in [-1,0] \\ \frac{3x^2-2x-1}{4}, & x \in [0,1] \end{cases}$$

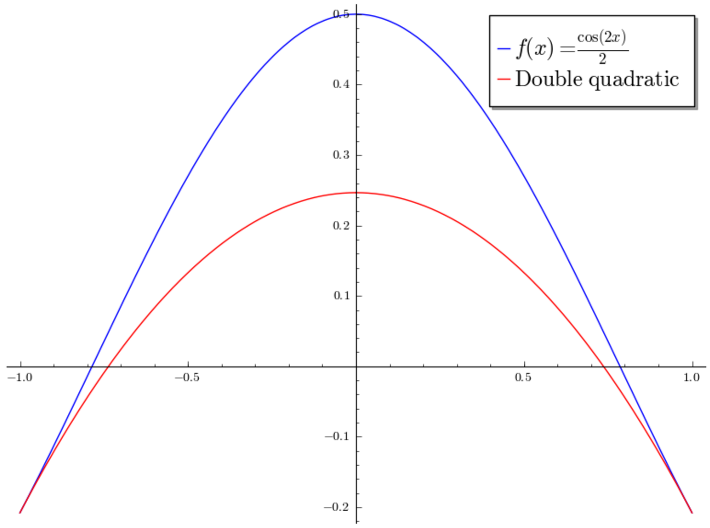

FIGURE 6.9. $s\left(\frac{\cos(2x)}{2}; x\right)$

The error bound for the double quadratic is,

Error. The error of these splines is given by,

$$e_2\,(\text{quasi-quadratic}; x) = \frac{1-x^2}{4}$$

$$e_2\,(\text{cubic}; x) = -\frac{x^4-2x^2+1}{x^2-4}$$

$$e_2\,(\text{double-quadratic}; x) \quad = \quad \begin{cases} -\frac{4x^4+4x^3-3x^2-2x+1}{4(x^2+2x-1)}, & x \in [-1,0] \\ -\frac{4x^4-4x^3-3x^2+2x+1}{4(x^2-2x-1)}, & x \in [0,1] \end{cases}$$

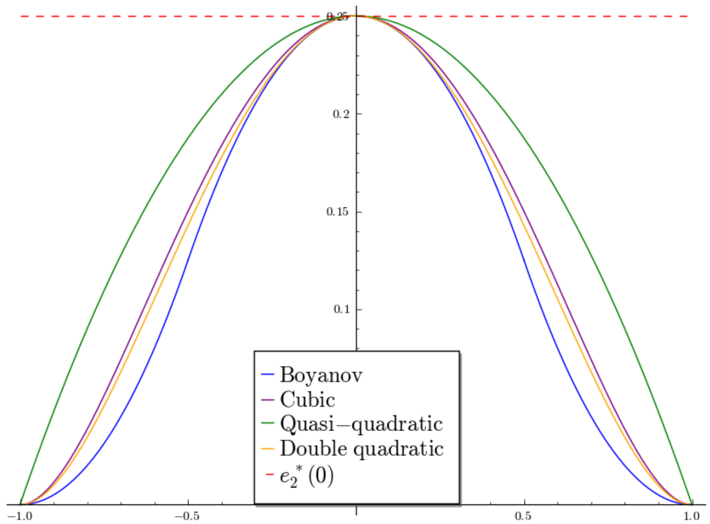

FIGURE 6.10. Error of optimal splines on W^2

W^3

For W^3, our optimal error bound is $e_3^*(0) = \frac{2-\sqrt{2}}{12}$.

TABLE 5. List of optimal splines on W^3

Description	Degree	Knots	Interpolation	Smoothness	Score
Boyanov	2	3	v,d,dd	1	-1
Quasi-quartic	4	0	v,d	2	0
Double cubic	3	1	v,d	1	1
Natural cubic	3	2	v,d,dd	1	-1

Boyanov. The formula for Boyanov's spline does not display well:

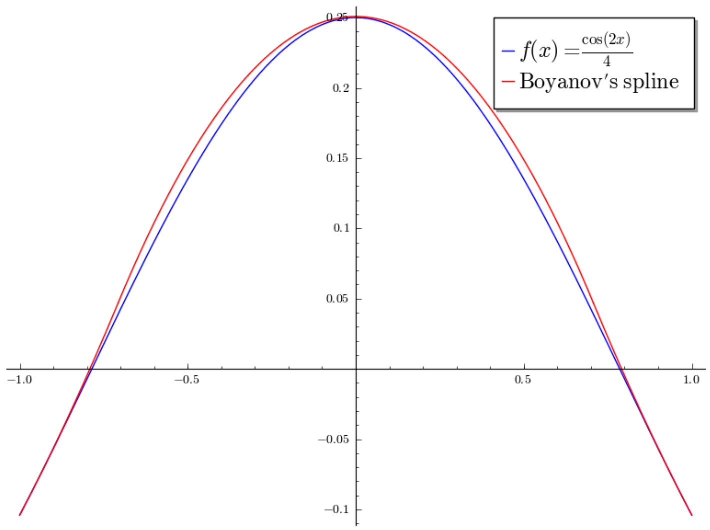

FIGURE 6.11. $S\left(\frac{\cos(2x)}{4};x\right)$

Quasi-quartic.

$$s\left(f;x\right)=\sum_{k=0}^{2}A_{k}f^{(k)}\left(-1\right)+B_{k}f^{(k)}\left(1\right)$$

where,

$$\begin{aligned}
A_0 &= \frac{x^3-3x+2}{4}\\
A_1 &= \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}-1\right)x^4+\sqrt{2}x^3-\left(3\sqrt{2}-2\right)x^2-\sqrt{2}x+2\sqrt{2}-1\right]\\
A_2 &= \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}-1\right)*x^4-2\left(\sqrt{2}-1\right)x^2+\sqrt{2}-1\right]\\
B_0 &= \frac{-x^3+3x+2}{4}\\
B_1 &= \frac{\sqrt{2}}{8}\left[(\sqrt{2}-1)x^4-\sqrt{2}x^3-\left(3\sqrt{2}-2\right)x^2+\sqrt{2}x+2\sqrt{2}-1\right]\\
B_2 &= \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}-1\right)x^4-2\left(\sqrt{2}-1\right)x^2+\sqrt{2}-1\right]
\end{aligned}$$

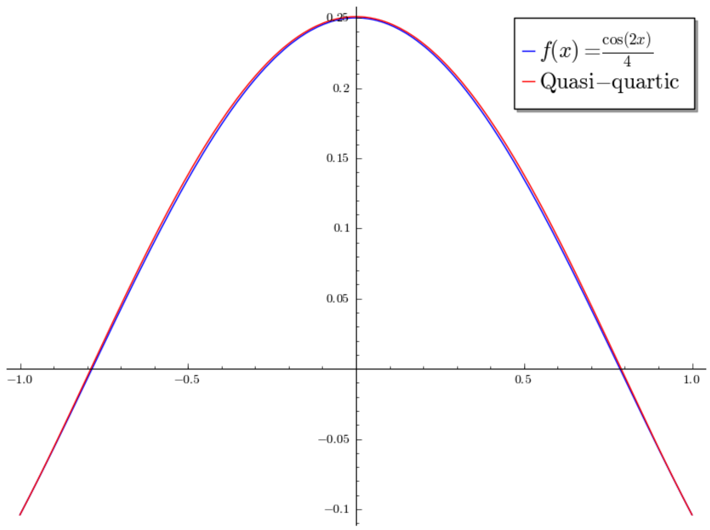

FIGURE 6.12. $s\left(\frac{\cos(2x)}{4};x\right)$

Double Cubic. The double cubic on W^3 is made from two separate cubics over $[-1,0]$ and $[0,1]$ respectively. We require that they join smoothly at $x=0$ where they both attain the value of Boyanov's spline.

$$s\left(f;x\right)=\sum_{k=0}^{2}A_k f^{(k)}\left(-1\right)+B_k f^{(k)}\left(1\right)$$

where,

$$A_0 = \begin{cases} \frac{-x^3+3x+2}{4}, & x\in[-1,0] \\ \frac{x^3-3x+2}{4}, & x\in[0,1] \end{cases}$$

$$A_1 = \begin{cases} -\frac{\sqrt{2}}{8}\left[\left(3\sqrt{2}-2\right)x^3+\left(4\sqrt{2}-3\right)x^2-\sqrt{2}x-2\sqrt{2}+1\right], & x\in[-1,0] \\ \frac{\sqrt{2}}{8}\left[\left(3\sqrt{2}-2\right)x^3-\left(4\sqrt{2}-3\right)x^2-\sqrt{2}x+2\sqrt{2}-1\right], & x\in[0,1] \end{cases}$$

$$A_2 = \begin{cases} -\frac{\sqrt{2}}{8}\left[2\left(\sqrt{2}-1\right)x^3+3\left(\sqrt{2}-1\right)x^2-\sqrt{2}+1\right], & x\in[-1,0] \\ \frac{\sqrt{2}}{8}\left[2\left(\sqrt{2}-1\right)x^3-3\left(\sqrt{2}-1\right)x^2+\sqrt{2}-1\right], & x\in[0,1] \end{cases}$$

$$B_0 = \begin{cases} \frac{x^3-3x+2}{4}, & x\in[-1,0] \\ \frac{-x^3+3x+2}{4}, & x\in[0,1] \end{cases}$$

$$B_1 = \begin{cases} \frac{\sqrt{2}}{8}\left[\left(\sqrt{2}-2\right)x^3+\left(4\sqrt{2}-3\right)x^2+\sqrt{2}x-2\sqrt{2}+1\right], & x\in[-1,0] \\ -\frac{\sqrt{2}}{8}\left[\left(\sqrt{2}-2\right)x^3-\left(4\sqrt{2}-3\right)x^2+\sqrt{2}x+2\sqrt{2}-1\right], & x\in[0,1] \end{cases}$$

$$B_2 = \begin{cases} -\frac{\sqrt{2}}{8}\left[2\left(\sqrt{2}-1\right)x^3+3\left(\sqrt{2}-1\right)x^2-\sqrt{2}+1\right], & x\in[-1,0] \\ \frac{\sqrt{2}}{8}\left[2\left(\sqrt{2}-1\right)x^3-3\left(\sqrt{2}-1\right)x^2+\sqrt{2}-1\right] & x\in[0,1] \end{cases}$$

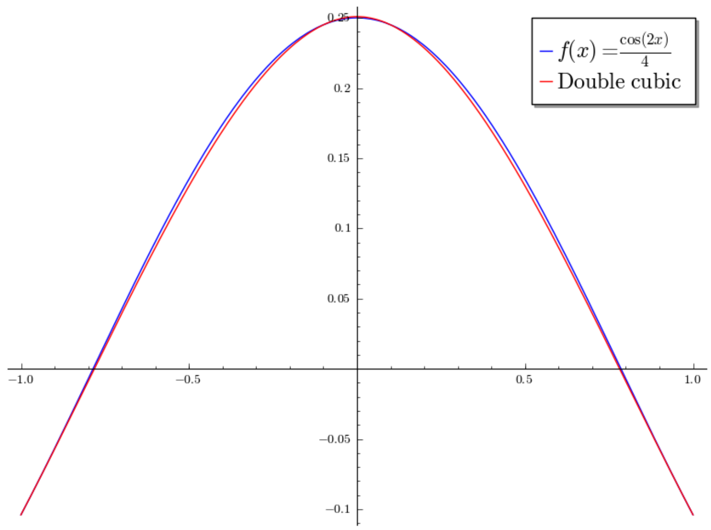

FIGURE 6.13. $s\left(\frac{\cos(2x)}{4}; x\right)$

Natural Cubic. The natural cubic has two split points, at a magic value,

$$\pm z = \frac{3\sqrt{2} - 4 + \sqrt{24\sqrt{2} - 30}}{8}$$

For the derivation of this value, see [7]. As a result, this spline has three pieces over $[-1, 1]$. The first piece interpolates up to the second derivative at $x = -1$, and the last piece interpolates up to the second derivative at $x = 1$. The middle piece interpolates the values and first derivatives at $\pm z$ for a C^1 function.

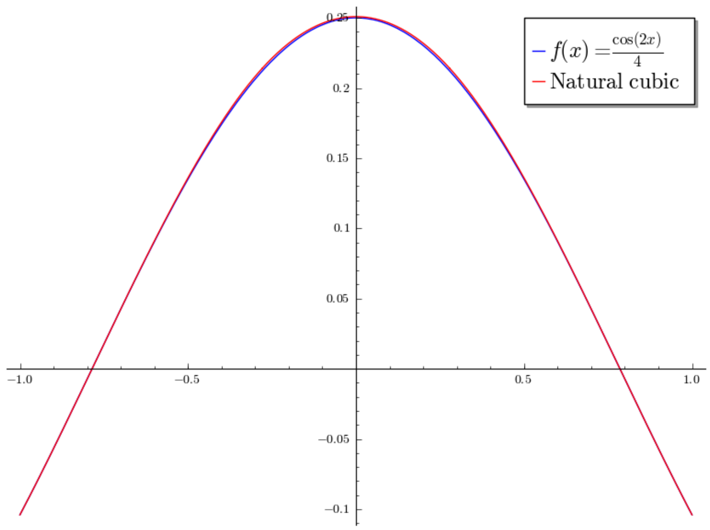

FIGURE 6.14. $s\left(\frac{\cos(2x)}{4}; x\right)$

Error. The expressions for error bounds on W^3 have already become unmanageable.

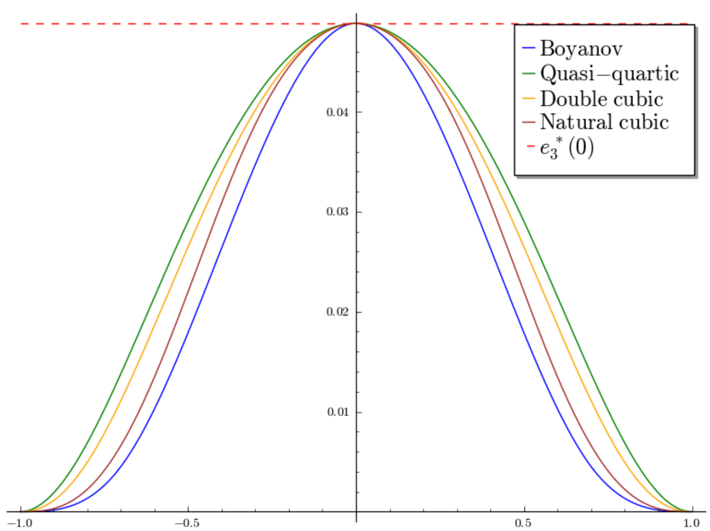

FIGURE 6.15. Error of optimal splines on W^3