APPROVAL SHEET

Title of Dissertation:	Positive operators, Z-operators, Lyapunov rank, and linear games on closed convex cones
Name of Candidate:	Michael Orlitzky Doctor of Philosophy, 2017
Dissertation and Abstract Approved:	Muddappa S. Gowda Professor Department of Mathematics and Statistics
Date Approved:	April 10, 2017

CURRICULUM VITAE

Name: Michael Joseph Orlitzky.

Degree and date to be conferred: Ph.D., 2017.

Secondary education: Baltimore Polytechnic Institute, Maryland, 2001.

Collegiate institutions attended:

(i) B.S. Mathematics, Towson University, 2009.

(ii) M.S. Applied and Industrial Mathematics, Towson University, 2012.

(iii) Ph.D. Applied Mathematics, University of Maryland Baltimore County, 2017.

Major: Applied Mathematics.

Professional publications:

(i) Michael Orlitzky and M. Seetharama Gowda. An improved bound for the Lyapunov rank of a proper cone. Optimization Letters, 10:11–17, 2016.

(ii) Michael Orlitzky. The Lyapunov rank of an improper cone. Optimization Methods and Software, 32(1):109–125, 2017.

Professional positions held:

(i) *Chief of Network Operations*, 2003–Present.
Viabit, LLC.
10534 York Road, Suite 202.
Hunt Valley, Maryland, 21030.

(ii) *Developer*, 2011–Present.
Sage Mathematics Project.
http://www.sagemath.org/

(iii) *Developer*, 2013–Present.
Gentoo Linux Project.
http://www.gentoo.org/

ABSTRACT

Title of dissertation: POSITIVE OPERATORS, Z-OPERATORS, LYAPUNOV RANK, AND LINEAR GAMES ON CLOSED CONVEX CONES

Michael Orlitzky, Doctor of Philosophy, 2017

Dissertation directed by: Professor Muddappa S. Gowda
Department of Mathematics and Statistics

Given a closed convex cone K with dual K^{*} in a finite-dimensional real Hilbert space, the linear operator L is positive on K if $L(K) \subseteq K$, and a **Z**-operator on K if $\langle L(x), s \rangle \leq 0$ for all $(x, s) \in K \times K^{*}$ with $\langle x, s \rangle = 0$. If both L and $-L$ are **Z**-operators on K, then L is *Lyapunov-like* on K. These concepts generalize (respectively) the nonnegative, **Z**, and diagonal matrices. They appear in various fields including dynamical systems, optimization, economics, and game theory. Our contribution is to extend results for proper cones to general closed convex cones.

We extend a result of Tam describing the dual of the cone of positive operators. We compute its largest linear subspace, and then use the exponential map to connect the positive and **Z**-operators. In particular, we show that both families share the same dimension and polyhedrality.

Motivated by optimization considerations, the *Lyapunov rank* of K is defined as the dimension of the space of all Lyapunov-like operators on K. We extend this concept from proper to closed convex cones, and provide an algorithm to efficiently compute the Lyapunov rank in that case. We further show that the Lyapunov rank of K is the dimension of the Lie algebra of the automorphism group of K in the general setting. We then improve an existing upper bound for the Lyapunov rank.

We introduce linear games on proper cones, and extend some results that Gowda and Ravindran formulated for self-dual cones. The more-general setting allows us to place linear games in the framework of conic programming.

POSITIVE OPERATORS, Z-OPERATORS, LYAPUNOV RANK, AND LINEAR GAMES ON CLOSED CONVEX CONES

by

Michael Orlitzky

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, Baltimore County in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2017

Advisory Committee:
Dr. Muddappa S. Gowda, Advisor
Dr. Osman Güler
Dr. Jinglai Shen
Dr. Roman Sznajder
Dr. Jiyuan Tao

Table of Contents

Chapter 1

Introduction

1.1 Background

Positive operators arose from the study of integral operators and matrices with nonnegative entries [1]. For our purposes, we define the positive operators on K by

$$\pi\left(K\right) \coloneqq \left\{L : V \to V \mid L \text{ is linear and } L\left(K\right) \subseteq K\right\}.$$

The famous Krein-Rutman theorem [25] connects positive operators to the theory of dynamical systems [34], to game theory [13], and more.

A $\mathbf{Z}$-matrix is a real square matrix whose off-diagonal entries are nonpositive. Equivalently, a $\mathbf{Z}$-matrix has the form $\lambda I - N$ where N is a nonnegative matrix (a positive operator on $\mathbb{R}^n_+$). Berman and Plemmons [5] cite a number of equivalent conditions for $\mathbf{Z}$-matrices, connecting them to many different areas.

Throughout the remainder of this chapter, V represents a finite-dimensional real Hilbert space. If K^* is the dual of K, then we define the *complementarity set*

$$C\left(K\right) \coloneqq \left\{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\right\}.$$

Such a set arises as optimality conditions in convex optimization [6, 7]. We say that L is a ***Z**-operator* on K [18] and we write $L \in \mathbf{Z}\left(K\right)$ if $\langle L\left(x\right), s\rangle \leq 0$ for all $(x, s) \in C\left(K\right)$. This definition reduces to that of a $\mathbf{Z}$-matrix when $K = \mathbb{R}^n_+$. These $\mathbf{Z}$-operators appear in dynamical systems [18], complementarity problems [18], game theory [15], economics, et cetera.

Schneider and Vidyasagar [35] found an important connection between the positive and **Z**-operators. We eventually extend it to closed convex cones.

Theorem (Schneider and Vidyasagar)**.** *If K is a proper cone in $\mathbb{R}^n$ and if A is a matrix in $\mathbb{R}^{n\times n}$, then $A \in \mathbf{Z}(K)$ if and only if $e^{-tA} \in \pi(K)$ for all $t \geq 0$.*

The set of all **Z**-operators contains an interesting subset, the set of Lyapunov-like operators. Rudolf et al. [33] investigated the possibility of expressing the condition $\langle x, s\rangle = 0$ in $C(K)$ as a system of equations $\langle L_i(x), s\rangle = 0$ for $i \geq 1$. Gowda and Tao [19] observed that the number of independent equations $\langle L_i(x), s\rangle = 0$ can be described in terms of the linear operators L such that $\langle L(x), s\rangle = 0$ for all $(x, s) \in C(K)$. These operators are related to the Lyapunov transformations in the theory of dynamical systems; hence they are called *Lyapunov-like* operators on K. The vector space of all Lyapunov-like operators on K is denoted by $\mathbf{LL}(K)$, and it is easy to see that $\mathbf{LL}(K) = -\mathbf{Z}(K) \cap \mathbf{Z}(K)$. The name *Lyapunov rank* is given to the dimension of $\mathbf{LL}(K)$, and the Lyapunov rank of K is denoted by $\beta(K)$.

Gowda and Tao connect $\mathbf{LL}(K)$ to the Lie algebra [3, 19] of the automorphism group of a proper cone K. We will extend their result to all closed convex cones.

Theorem (Gowda and Tao)**.** *If K is a proper cone and if L is linear, then $L \in \mathbf{LL}(K)$ if and only if L belongs to the Lie algebra of the automorphism group of K.*

Recent work has focused on bounding and computing the Lyapunov rank. Rudolf et al. [33] computed the Lyapunov rank of the moment cone. Gowda and Tao [19] derived results for polyhedral and symmetric cones. Gowda, Sznajder, and

Tao [17] investigated the completely-positive and copositive cones, and Gowda and Trott [20] have considered special Bishop-Phelps cones. Gowda and Tao were able to derive the upper bound $\beta(K) \leq n^2 - n$ for a proper cone K in an n-dimensional space. We improve that bound to $(n-1)^2$.

We analyze these operators in the context of game theory as well. A classical zero-sum matrix game [40] involves a matrix $A \in \mathbb{R}^{n \times n}$. Two players choose "strategies" x and y, respectively, from the unit simplex $\Delta \coloneqq \left\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\right\}$, and the "payoff" to the first player is $y^T A x$. The existence of optimal strategies for both players follows from a general min-max theorem [23], and the value of the game is defined to be the payoff associated with the optimal strategies.

Gowda and Ravindran [15] have shown that we can generalize a zero-sum matrix game to a *linear game* whose payoff function is based on a linear operator and whose strategy sets are compact convex subsets of a self-dual cone $K = K^*$. The min-max theorem still guarantees the existence of optimal strategies, and much of the theory carries over. We are able to extend these results, allowing for $K \neq K^*$ and for a wider range of strategies. The choice of operator affects the properties of the game, and we consider games related to Lyapunov-like and **Z**-operators. Linear games are then placed in the framework of conic programming.

1.2 Organization of results

We begin Section 2.1 by computing the generators of $\pi(K)^*$. Immediately we have an algorithm for computing $\pi(K)$ itself when K is polyhedral. The dual generators let us characterize the dimension and lineality of $\pi(K)$, from which we conclude that $\pi(K)$ is proper if and only if K is proper. We extend a result of Tam [38] to show that $\pi(K)$ is polyhedral if and only if K is polyhedral. Section 2.2 is similar in spirit to Section 2.1. We compute the generators of $\mathbf{Z}(K)^*$, and obtain an algorithm to compute $\mathbf{Z}(K)$ for a polyhedral K. The dimension of $\mathbf{Z}(K)$ is shown to equal that of $\pi(K)$. As with $\pi(K)$, it turns out that $\mathbf{Z}(K)$ is polyhedral if and only if K is polyhedral. Section 2.3 is devoted to Lyapunov-like operators. We extend many of the basic results for proper cones to the closed and convex case. We are able to show that K is "perfect" in V if and only if $\beta(K) \geq \dim(V)$.

Section 3.1 begins by introducing the cone-space pair notation. The new notation is used in Section 3.2 to derive a formula for the Lyapunov rank of a closed convex cone in terms of a proper subcone. Gowda and Tao [19] showed that $\beta(K) \neq \dim(V) - 1$, and we show that the same is true for improper cones. In Section 3.3, an efficient algorithm is derived to compute $\beta(K)$ for polyhedral K. Finally in Section 3.4, we extend a result of Schneider and Vidyasagar [35] to show that the exponential function connects $\mathbf{Z}(K)$ to $\pi(K)$. That fact is then used to show that $\mathbf{LL}(K)$ is the Lie algebra of the automorphism group of K.

Gowda and Tao [19] proved that if K is a proper cone in an n-dimensional space, then $\beta(K) \leq n^2 - n$. Chapter 4 is devoted entirely to improving that bound, and we show that $\beta(K) \leq (n-1)^2$ in the same setting.

Section 5.1 introduces the concept of a linear game. In Section 5.2 we prove some basic facts about linear games and their values; in particular we state necessary and sufficient conditions for optimality. Then in Section 5.3, we define completely-mixed games and show that they have unique solutions under certain conditions. Section 5.4 considers the case where the payoff operator is Lyapunov-like, Stein-like, or a **Z**-operator on the underlying cone. Section 5.5 concludes by showing that every linear game can be solved by a conic program.

1.3 Standard terminology

Let V and W be finite-dimensional real Hilbert spaces. The set of all linear operators from V to W forms a vector space which we denote by $\mathcal{B}(V, W)$. We abbreviate $\mathcal{B}(V, V)$ by $\mathcal{B}(V)$. If $L \in \mathcal{B}(V, W)$ is invertible and preserves inner products, then L is an *isometry*. If $L(X) = Y$ under some isometry L, then X and Y are *isometric* and we write $X \cong Y$. If $L \in \mathcal{B}(V)$ is invertible and $L(X) = X$, then L is an automorphism of X and we write $L \in \operatorname{Aut}(X)$.

Any $L \in \mathcal{B}(V, W)$ has an adjoint $L^{*} \in \mathcal{B}(W, V)$ such that $\langle L(x), y\rangle = \langle x, L^{*}(y)\rangle$ for all $x \in V$ and $y \in W$. Given two elements x and s in V, we define $s \otimes x$ to be the operator $y \mapsto \langle x, y\rangle s$ on V. For subsets S and X of V, we will write $S \otimes X \coloneqq \{s \otimes x \mid s \in S, x \in X\}$. The adjoint of $s \otimes x$ is $x \otimes s$, and $s \otimes L^{*}(x) = (s \otimes x) \circ L$ is the composition of the operators $s \otimes x$ and $L \in \mathcal{B}(V)$. The identity operator on V is $\mathrm{id}_V \in \mathcal{B}(V)$. In $\mathbb{R}^n$, the identity matrix of the appropriate size is denoted by I.

If $\sigma(L)$ denotes the spectrum of L, we define the trace operator on $\mathcal{B}(V)$ by $\operatorname{trace}(L) \coloneqq \sum_{\lambda \in \sigma(L)} \lambda$. Then $\langle L_1, L_2\rangle \coloneqq \operatorname{trace}(L_1 \circ L_2^{*})$ is our inner product on spaces of operators. Later we use the fact that $\operatorname{trace}(s \otimes x) = \operatorname{trace}(x \otimes s) = \langle x, s\rangle$.

The *convex hull* of a nonempty subset X of V is

$$\operatorname{conv}(X) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \geq 0,\ m \in \mathbb{N},\ \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

The *affine hull* is likewise defined,

$$\operatorname{aff}(X) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \in \mathbb{R},\ m \in \mathbb{N},\ \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

The topological *interior* of a set X is $\operatorname{int}(X)$, its *closure* is $\operatorname{cl}(X)$, and its *boundary* is $\operatorname{bdy}(X)$. Let $B_\epsilon(x)$ denote an open ϵ-ball at x. If X is convex, then its *relative boundary* is the set of all $x \in \operatorname{cl}(X)$ such that for any $\epsilon > 0$, the set $B_\epsilon(x) \cap \operatorname{aff}(X)$ intersects both X and $V \setminus X$ nontrivially.

If W is a subspace of V, then the *orthogonal complement* of W is another subspace of V defined by $W^\perp \coloneqq \{ y \in V \mid \langle x, y \rangle = 0 \text{ for all } x \in W \}$, and V has direct sum decomposition $V = W \oplus W^\perp$. If $x = x_1 + x_2$ for $x_1 \in W$ and $x_2 \in W^\perp$, then $\operatorname{proj}(W, x_1 + x_2) = x_1$ defines the *orthogonal projection* of $x \in V$ onto the subspace W. For $x \in V$, we abbreviate $x^\perp \coloneqq \operatorname{span}(\{x\})^\perp$. The direct sum of two orthogonal sets X and S will be denoted by $X \obot S$; for example, it is appropriate to write $V = W \obot W^\perp$ when W is a subspace of V.

The real n-space $\mathbb{R}^n$ comes with the usual inner product and standard basis $\mathbf{e} \coloneqq \{e_1, e_2, \ldots, e_n\}$. In $\mathbb{R}^{n \times n}$ we use the standard basis $\left\{ E_{ij} = e_i e_j^T \;\middle|\; 1 \leq i, j \leq n \right\}$. The spaces $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n^2}$ are isometric under the inverse operations $\operatorname{vec}(\cdot)$ and $\operatorname{mat}(\cdot)$ which transform a matrix into a long vector and vice-versa. The cardinality of a set X is $\operatorname{card}(X)$; for example, $\operatorname{card}(\mathbf{e}) = n$.

1.4 Cone terminology

Definition 1. A nonempty subset K of V is a *cone* if $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V.

Definition 2. The *conic hull* of a nonempty subset X of V is

$$\operatorname{cone}(X) := \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \geq 0,\ m \in \mathbb{N} \right\}.$$

Clearly $\operatorname{cone}(X)$ is a convex cone. If X is finite, then $\operatorname{cone}(X)$ is closed [31].

Definition 3. If $\operatorname{cone}(G) = K$, then G *generates* K and the elements of G are *generators* of K. If a finite set generates K, then K is *polyhedral*.

Definition 4. The *dimension* of $K \subseteq V$ is $\dim(K) := \dim(\operatorname{span}(K))$. Its *codimension* is $\operatorname{codim}(K) := \dim(V) - \dim(K)$. A convex cone K is *solid* if $\operatorname{span}(K) = V$.

If S and X are subsets of V, then $\dim(S \otimes X) = \dim(S)\dim(X)$ [32].

Definition 5. The *lineality space* of a convex cone K is $\operatorname{linspace}(K) := -K \cap K$. Its *lineality* is $\operatorname{lin}(K) := \dim(\operatorname{linspace}(K))$ and K is *pointed* if $\operatorname{lin}(K) = 0$.

Definition 6. A pointed, solid, and closed convex cone is *proper*.

Definition 7. A nonempty convex subset F of a convex cone K is a *face* of K if $x, y \in K$ and $\alpha x + (1 - \alpha) y \in F$ for $0 < \alpha < 1$ together imply that $x, y \in F$. If in addition we have $\dim(F) = 1$, then F is an *extreme ray* of K. The *extreme directions* of K are representatives of its extreme rays defined by,

$$\operatorname{Ext}(K) := \{x \mid x \text{ belongs to an extreme ray of } K \text{ and } \|x\| = 1\}.$$

If K is a closed, convex, and pointed cone, then $K = \operatorname{cone}(\operatorname{Ext}(K))$ by a conic version of the Krein-Milman theorem.

Definition 8. If K is a subset of V, then the *dual cone* K^* of K is

$$K^* := \{y \in V \mid \forall x \in K,\ \langle x, y \rangle \geq 0\}.$$

The dual K^* is always a closed convex cone. If K is a closed convex cone, then $(K^*)^* = K$. If $K = K^*$, we say that K is *self-dual*.

Any proper cone K orders its ambient space by $x \succcurlyeq_K y$ if and only if $x - y \in K$. One writes $x \succ_K y$ to indicate $x - y \in \operatorname{int}(K)$. Often, the cone K is understood and the subscript is omitted. Inequality with respect to the dual cone K^* is abbreviated:

$$x \succcurlyeq y \iff x - y \in K; \quad x \succ y \iff x - y \in \operatorname{int}(K)$$

$$x \overset{*}{\succcurlyeq} y \iff x - y \in K^*; \ x \overset{*}{\succ} y \iff x - y \in \operatorname{int}(K^*).$$

We will make use of the ordering's transitivity; if $x \succcurlyeq_K y \succ_K 0$, then $x \succ_K 0$.

Definition 9. A self-dual cone K is *symmetric* if the automorphism group $\operatorname{Aut}(K)$ acts transitively on $\operatorname{int}(K)$. That is, if for all $x \in \operatorname{int}(K)$ and all $y \in \operatorname{int}(K)$, there exists an $L \in \operatorname{Aut}(K)$ such that $L(x) = y$.

A few cones will appear often. We collect them here for convenience.

Definition 10. The *nonnegative orthant* in $\mathbb{R}^n$ is

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Definition 11. The *symmetric positive semidefinite cone* is

$$\mathcal{S}^n_+ := \left\{A \in \mathbb{R}^{n \times n} \;\middle|\; A^T = A \text{ and } \langle Ax, x \rangle \geq 0 \text{ for all } x \in \mathbb{R}^n\right\}.$$

Definition 12. The *Hermitian positive semidefinite cone* is

$$\mathcal{H}_+^n := \left\{ A \in \mathbb{C}^{n \times n} \;\middle|\; A^* = A \text{ and } \langle Ax, x \rangle \geq 0 \text{ for all } x \in \mathbb{C}^n \right\}.$$

1.5 Some classes of linear operators

Definition 13. An $L \in \mathcal{B}(V)$ is a *positive operator* on a set $K \subseteq V$ if $L(K) \subseteq K$. The set of all such operators is denoted by $\pi(K)$. If K is a closed convex cone, then

$$L \in \pi(K) \iff \langle L(x), s \rangle \geq 0 \text{ for all } (x, s) \in K \times K^*.$$

The requisite property of a **Z**-operator is similar, but it need only hold for pairs of orthogonal vectors in $K \times K^*$.

Definition 14. The *complementarity set* of K is

$$C(K) := \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

In many optimization problems, membership in the complementarity set is a condition for optimality. One strategy for solving those problems is to replace the single equation $\langle x, s \rangle = 0$ by an equivalent system of two or more independent equations. Some examples are given in Section 1.6.

Definition 15. The operator $L \in \mathcal{B}(V)$ is a ***Z**-operator* on a set $K \subseteq V$ if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K).$$

By $\mathbf{Z}(K)$ we denote the set of all **Z**-operators on K.

When $K = \mathbb{R}_+^n$, the complementarity set $C\left(\mathbb{R}_+^n\right)$ contains all pairs of distinct standard basis vectors. The requirement on $\mathbf{Z}\left(\mathbb{R}_+^n\right)$ gives rise to matrices whose off-diagonal elements are nonpositive—the **Z**-matrices.

Definition 16. If $K = \operatorname{cone}(G_1)$ and $K^* = \operatorname{cone}(G_2)$, then we refer to the set $C(K) \cap (G_1 \times G_2)$ as a *discrete complementarity set* of K.

If K is polyhedral, it has a finite discrete complementarity set. Our next example takes advantage of the fact (proved later, in Proposition 19) that $L \in \mathbf{Z}(K)$ if and only if it satisfies Definition 15 on a discrete complementarity set.

Example 1. If $K = \mathbb{R}^n_+$, then $K = \operatorname{cone}(\mathbf{e})$ and thus

$$C(K) \cap (\mathbf{e} \times \mathbf{e}) = \{(e_i, e_j) \in \mathbf{e} \times \mathbf{e} \mid i \neq j\}.$$

If $L \in \mathbf{Z}\left(\mathbb{R}^n_+\right)$, then the condition $\langle L(e_i), e_j \rangle \leq 0$ associated with each (e_i, e_j) implies that $L_{ji} \leq 0$. Since i and j range over all indices where $i \neq j$, all of the off-diagonal entries of L are nonpositive. This is also sufficient for $L \in \mathbf{Z}(K)$, since every such L will have $\langle L(e_i), e_j \rangle \leq 0$ when $i \neq j$.

Definition 17. The operator $L \in \mathcal{B}(V)$ is *Lyapunov-like* on a set $K \subseteq V$ if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

By $\mathbf{LL}(K)$ we denote the set of all Lyapunov-like operators on K.

Lyapunov's theorem [12, 14] characterizes the stability of a dynamical system in terms of the Lyapunov transformation $X \mapsto AX + XA^*$. Damm [8] showed that the Lyapunov-like operators are nothing other than Lyapunov transformations when K is $\mathcal{S}^n_+$ or $\mathcal{H}^n_+$. The set $\mathbf{LL}(K)$ is a vector space and $\mathbf{LL}(K) = \operatorname{linspace}(\mathbf{Z}(K))$. In other words, $L \in \mathbf{LL}(K)$ if and only if both $L \in \mathbf{Z}(K)$ and $-L \in \mathbf{Z}(K)$.

Example 2. If $K = \mathbb{R}^n_+$, then $L \in \mathbf{Z}(K)$ if and only if the off-diagonal entries of L are nonpositive. But $-L \in \mathbf{Z}(K)$ if and only if its off-diagonals are nonnegative.

Thus $L \in \mathbf{LL}(K)$ if and only if L is a diagonal matrix. The space of all diagonal matrices is an n-dimensional subspace of $\mathbb{R}^{n \times n}$, so $\dim(\mathbf{LL}(K)) = n$.

Finding and counting linearly-independent Lyapunov-like operators is an interesting problem. Rudolf et al. [33] introduced the Lyapunov rank under the name *bilinearity rank* and computed it for a few cones of interest. Their work is continued by others [17, 19, 20, 27, 28]. Each Lyapunov-like operator in a linearly-independent set produces an additional equation from the complementarity condition $(x, s) \in C(K)$. The number of independent equations that we can obtain therefore depends on the dimension of $\mathbf{LL}(K)$. In a sense, $\dim(\mathbf{LL}(K))$ measures how easy it is to solve an optimization problem by solving its complementarity condition.

Definition 18. The *Lyapunov rank* of K is $\beta(K) \coloneqq \dim(\mathbf{LL}(K))$.

Stein's theorem [14, 36] is a discrete analogue of Lyapunov's theorem in continuous dynamical systems; it describes the so-called Stein transformations [18] that correspond to the Lyapunov transformations in continuous systems. We generalize the Stein transformations to Stein-like operators [15] in the same way that we have generalized the Lyapunov transformations to Lyapunov-like operators.

Definition 19. The operator $L \in \mathcal{B}(V)$ is *Stein-like* on a set $K \subseteq V$ if $L = \mathrm{id}_V - A$ for some $A \in \mathrm{cl}(\mathrm{Aut}(K))$.

Example 3. If $K = \mathcal{S}^n_+$, then $\mathrm{Aut}(K)$ consists of operators of the form $X \mapsto AXA^*$ where $A \in \mathbb{R}^{n \times n}$ is invertible [17]. Since the invertible matrices are dense in $\mathbb{R}^{n \times n}$, its closure is $\mathrm{cl}(\mathrm{Aut}(K)) = \{X \mapsto AXA^* \mid A \in \mathbb{R}^{n \times n}\}$. Stein's theorem [36] concerns operators of the form $X \mapsto X - AXA^*$, namely Stein-like operators on $K = \mathcal{S}^n_+$.

Example 4. If $K = \mathcal{H}_+^n$, then we can replace $\mathbb{R}$ by $\mathbb{C}$ in Example 3 to conclude that Stein transformations are Stein-like on $\mathcal{H}_+^n$ and vice-versa.

Our last family of operators arises in linear complementarity problems. The problem in Example 5 is feasible for all $q \in \mathbb{R}^n$ if and only if M is an **S**-matrix [7]; that is, if $M(d) \in \operatorname{int}\left(\mathbb{R}_+^n\right)$ for some $d \in \operatorname{int}\left(\mathbb{R}_+^n\right)$. The **S**-operators are a straightforward generalization of **S**-matrices to cones other than the nonnegative orthant.

Definition 20. The operator $L \in \mathcal{B}(V)$ is an ***S**-operator* on a set $K \subseteq V$ if there exists a $d \in \operatorname{int}(K)$ such that $L(d) \in \operatorname{int}(K)$. The set of all **S**-operators on K is denoted by $\mathbf{S}(K)$.

Duality is often reflected in these families; the following fact is easily proved.

Proposition 1. *If K is a closed convex cone, then $L \in \pi(K)$ if and only if $L^* \in \pi(K^*)$. Analogous statements hold for $\mathbf{Z}(K)$, $\mathbf{LL}(K)$, and $\operatorname{Aut}(K)$.*

1.6 Examples

Example 5. For $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, the linear complementarity problem is to

$$\begin{aligned} \text{find} \quad & x \geq 0 \\ \text{such that} \quad & q + Mx \geq 0 \\ \text{and } & (q+Mx)^T x = 0. \end{aligned}$$

If $s \coloneqq q + Mx$ and $K \coloneqq \mathbb{R}^n_+ = K^*$, then this can be rewritten as

$$\left.\begin{aligned} & x \in K \\ & s \in K^* \\ & \langle x, s \rangle = 0 \end{aligned}\right\} (x, s) \in C(K). \tag{1.1}$$

In other words, the problem description is captured by the complementarity set $C(K)$. We note that $\langle x, s \rangle = 0$ can be rewritten as a system of n equations,

$$x_1 s_1 = 0; \quad x_2 s_2 = 0; \quad \cdots; \quad x_n s_n = 0. \tag{1.2}$$

In this case, the system is easier to solve than the single equation $\langle x, s \rangle = 0$.

Example 6. A linear program [6] is an optimization problem with a linear objective function and some linear constraints. In the *primal* problem, we are asked to

$$\begin{aligned} \text{minimize} \quad & \langle b, x \rangle \\ \text{subject to} \ & L(x) \geq c \\ & x \geq 0. \end{aligned}$$

This problem has an associated *dual* problem, to

$$\begin{aligned} \text{maximize} \quad & \langle c, y \rangle \\ \text{subject to} \ & L^*(y) \leq b \\ & y \geq 0. \end{aligned}$$

The dual optimal value exists and equals that of the primal under certain conditions. If $(\bar{x}, \bar{y})$ is a pair of solutions to the primal and dual problems, respectively, then necessarily $\langle L\bar{x} - c, \bar{y}\rangle = 0$. This requirement is called "complementary slackness" and it amounts to $(L\bar{x} - c, \bar{y}) \in C\left(\mathbb{R}^n_+\right)$.

It is known [19] that $K = \mathbb{R}^n_+$ has Lyapunov rank n. This is reflected in the fact that the condition $\langle x, s\rangle = 0$ in (1.1) can be rewritten as a system of n equations (1.2). By combining that system with the n equations $s = q + Mx$, we obtain $2n$ equations in the $2n$ variables x_i, s_j for $1 \le i, j \le n$.

These examples extend to cones other than the nonnegative orthant, and solving that type of problem is one of the goals of Chapter 5. Linear programs and certain matrix games are known to be equivalent. We demonstrate a similar connection between *conic programming* and a new concept called a *linear game*.

Chapter 2

Special linear operators on closed convex cones

The theory of positive, **Z**, and Lyapunov-like operators has so far concentrated on *proper* cones. However, there are no technical impediments that require a positive, **Z**, or Lyapunov-like operator to be defined on a proper cone. In this chapter, we extend the theory of these operators to any closed convex cone. We will see that the added generality reveals some connections between the positive and **Z**-operators.

The next three results are well-known and are given as Rockafellar's [31] Theorem 14.6, Corollary 16.4.2, and Corollary 19.2.2, respectively.

Proposition 2. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then K is pointed if and only if K^{*} is solid. Moreover* $\operatorname{linspace}(K) = \operatorname{span}(K^{*})^{\perp}$ *and* $\operatorname{lin}(K) = \operatorname{codim}(K^{*})$.

Proposition 3. *If K_1 and K_2 are convex cones in a finite-dimensional real Hilbert space, then $(K_1 + K_2)^{*} = K_1^{*} \cap K_2^{*}$ and $(\operatorname{cl}(K_1) \cap \operatorname{cl}(K_2))^{*} = \operatorname{cl}(K_1^{*} + K_2^{*})$. If K_1 and K_2 are closed, then it follows that the dual of $\operatorname{cl}(K_1^{*} + K_2^{*})$ is $K_1 \cap K_2$.*

Proposition 4. *The dual of a polyhedral convex cone is polyhedral.*

Sometimes we will need to find elements $x \in K$ and $s \in K^{*}$ having $\langle x, s \rangle > 0$. If K is solid, we can choose x in its interior and $s \in K^{*}$ nonzero but otherwise arbitrarily. That approach fails when K has no interior, motivating the following consequence of Proposition 2. Its proof is immediate from $\operatorname{linspace}(K) = \operatorname{span}(K^{*})^{\perp}$.

Corollary 1. *If K is a closed convex cone in a finite-dimensional real Hilbert space and if $x \in K \setminus \operatorname{linspace}(K)$, then there exists an $s \in K^*$ such that $\langle x, s \rangle > 0$.*

We will also need to find nontrivial orthogonal pairs of vectors in K and its dual. We restate Fenchel's [11] Corollary 2 in more convenient terms.

Proposition 5. *If K is a closed convex cone and if $x \in \operatorname{bdy}(K)$, then there exists a nonzero $s \in K^*$ such that $\langle x, s \rangle = 0$.*

2.1 Positive operators

The set of all positive operators on a closed convex cone K itself forms a closed convex cone. The three properties—that $\pi(K)$ is closed, convex, and a cone—are easy to verify and depend on the same properties of K.

Proposition 6. *If K is a closed convex cone, then so is $\pi(K)$.*

If K is proper, then both $\pi(K)$ and its dual are proper [35]. To determine if some linear operator belongs to $\pi(K)$, it suffices to check positivity on a generating set of K. This follows easily from the linearity of the operator.

Proposition 7. *If $K = \operatorname{cone}(G)$ in a finite-dimensional real Hilbert space V and if $L \in \mathcal{B}(V)$, then $L \in \pi(K)$ if and only if $L(G) \subseteq K$.*

Tam found a simple expression for the generators of the dual of $\pi(K)$ when K is proper [38]. He uses the fact that $\operatorname{cone}(K^* \otimes K)$ is closed to prove that

$$\pi(K)^* = \operatorname{cone}(K^* \otimes K) \text{ if } K \text{ is proper.}$$

This fact requires a new proof when K is merely closed and convex. In that setting, we define $\mathcal{T} \coloneqq \operatorname{cone}(K^* \otimes K)$ to be the cone of Tam's generators. Note that any $\phi \in \mathcal{T}$ can be written $\phi = \sum_{i=1}^{m} s_i \otimes x_i$ without the scalar factors, since they can be absorbed into $s_i \otimes x_i$. Our first step is to decompose $\mathcal{T}$ into two components [31].

Proposition 8. *If K is a convex cone in a finite-dimensional real Hilbert space, then K has an orthogonal direct sum decomposition into two convex cones,*

$$K = K \cap \operatorname{linspace}(K)^{\perp} \oplus \operatorname{linspace}(K).$$

Its first factor $K \cap \operatorname{linspace}(K)^{\perp} = \operatorname{proj}\left(\operatorname{linspace}(K)^{\perp}, K\right)$ is pointed.

We will ultimately apply this decomposition to $\mathcal{T} \coloneqq \operatorname{cone}(K^* \otimes K)$. The crux of our argument is that $\mathcal{T}$ is closed even if K is not proper. This next result reduces the burden of proof to the pointed component of $\mathcal{T}$ obtained from Proposition 8.

Proposition 9. *If K is a convex cone in a finite-dimensional real Hilbert space, then K is closed if and only if $\operatorname{proj}\left(\operatorname{linspace}(K)^{\perp}, K\right)$ is closed.*

Proof. If K is closed, then $\operatorname{proj}\left(\operatorname{linspace}(K)^{\perp}, K\right) = K \cap \operatorname{linspace}(K)^{\perp}$ is the intersection of two closed sets by Proposition 8.

If $\operatorname{proj}\left(\operatorname{linspace}(K)^{\perp}, K\right)$ is closed, then Proposition 8 shows that any convergent sequence $(v+w)_n$ contained in K will decompose into orthogonal parts $v_n + w_n$ where $(v)_n$ converges in $\operatorname{proj}\left(\operatorname{linspace}(K)^{\perp}, K\right)$ and $(w)_n$ converges in $\operatorname{linspace}(K)$. Thus $(v+w)_n$ converges to a point of K. $\square$

To exploit Proposition 8, we require the generators of its pointed component.

Proposition 10. *If* $K = \operatorname{cone}(G)$ *in a finite-dimensional real Hilbert space* V *and if* W *is a subspace of* V*, then* $\operatorname{proj}(W, G)$ *generates* $\operatorname{proj}(W, K)$*.*

Proof. Since $\operatorname{proj}(W, K) = \{\operatorname{proj}(W, \sum_{i=1}^{m} \alpha_i g_i) \mid \alpha_i \geq 0,\ g_i \in G,\ m \in \mathbb{N}\}$, the linearity of the projection gives $\operatorname{proj}(W, K) = \operatorname{cone}(\operatorname{proj}(W, G))$. $\square$

We want to apply these results to $\mathcal{T}$, but we are missing two pieces of information: we lack a description of $\operatorname{linspace}(\mathcal{T})^{\perp}$, and also the form of the projected generators. We can compute $\operatorname{linspace}(\mathcal{T})$ explicitly, but we need an easy result first.

Proposition 11. *If* K *is a closed convex cone in a finite-dimensional real Hilbert space and if* $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$*, then* $\mathcal{T} \subseteq \pi(K^*)$*.*

Proof. If $L = \sum_{i=1}^{m} s_i \otimes x_i \in \mathcal{T}$ and $t \in K^*$, then $L(t) = \sum_{i=1}^{m} \langle x_i, t \rangle s_i \in K^*$. $\square$

Proposition 12. *If* K *is a closed convex cone in a finite-dimensional real Hilbert space and if* $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$*, then* $\operatorname{linspace}(\mathcal{T}) = U_1 + U_2$ *where*

$$U_1 \coloneqq \operatorname{span}(K^* \otimes \operatorname{linspace}(K)); \quad U_2 \coloneqq \operatorname{span}(\operatorname{linspace}(K^*) \otimes K).$$

Proof. Any $u \in U_1 + U_2$ is a linear combination of terms like $s_i \otimes x_i$ where $(x_i, s_i) \in K \times K^*$ and either $x_i \in \operatorname{linspace}(K)$ or $s_i \in \operatorname{linspace}(K^*)$. Thus we can write $\pm u = \sum \pm\alpha_i (s_i \otimes x_i)$ for $\pm\alpha_i \in \mathbb{R}$. Group the coefficient $\pm\alpha_i$ with the appropriate lineality space element to see that $\pm u \in \mathcal{T}$, or that $u \in \operatorname{linspace}(\mathcal{T})$.

For the other inclusion, suppose that $L \coloneqq \sum_{i=1} s_i \otimes x_i \in \operatorname{linspace}(\mathcal{T})$ with $(x_i, s_i) \in K \times K^*$. Then both $L \in \mathcal{T}$ and $-L \in \mathcal{T}$, so $-L + \sum_{i>1} s_i \otimes x_i = -(s_1 \otimes x_1)$ is the sum of two elements of $\mathcal{T}$, and is therefore in $\mathcal{T}$ itself. Conclude that $s_1 \otimes x_1 \in \operatorname{linspace}(\mathcal{T})$. Suppose $x_1 \notin \operatorname{linspace}(K)$ and $s_1 \notin \operatorname{linspace}(K^*)$ so that by Corollary 1, we obtain a $t_1 \in K^*$ such that $\langle x_1, t_1\rangle > 0$. In that case, we apply $\pm(s_1 \otimes x_1) \in \mathcal{T}$ to t_1 and use Proposition 11 to show that $\pm s_1 \in K^*$. Then $s_1 \in \operatorname{linspace}(K^*)$ is a contradiction, so we conclude that either $x_1 \in \operatorname{linspace}(K)$ or $s_1 \in \operatorname{linspace}(K^*)$, and thus $s_1 \otimes x_1 \in U_1 + U_2$. Reason similarly about the other $s_i \otimes x_i$ with $i \neq 1$ to show that $L \in U_1 + U_2$. $\square$

We can now compute $\operatorname{lin}(\mathcal{T})$ and subsequently find $\operatorname{linspace}(\mathcal{T})^\perp$.

Proposition 13. *If K is a closed convex cone in a finite-dimensional real Hilbert space V and if $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$, then*

$$\operatorname{lin}(\mathcal{T}) = \operatorname{lin}(K)\dim(K^*) + \dim(K)\operatorname{lin}(K^*) - \operatorname{lin}(K)\operatorname{lin}(K^*).$$

Proof. Use Proposition 12 and apply the dimension formula to $\operatorname{linspace}(\mathcal{T})$,

$$\dim(\operatorname{linspace}(\mathcal{T})) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Clearly $\dim(U_1) = \operatorname{lin}(K)\dim(K^*)$ and $\dim(U_2) = \dim(K)\operatorname{lin}(K^*)$. And since $\operatorname{linspace}(K^*) \otimes \operatorname{linspace}(K) \subseteq U_1 \cap U_2$, we have $\dim(U_1 \cap U_2) \geq \operatorname{lin}(K)\operatorname{lin}(K^*)$. Yet any $u_1 \in U_1$ and $u_2 \in U_2$ satisfy

$$u_1\left(\operatorname{linspace}(K)^\perp\right) = \{0\} \text{ and } u_2(V) \subseteq \operatorname{linspace}(K^*),$$

so $\dim(U_1 \cap U_2) \leq \dim(\mathcal{B}(\operatorname{linspace}(K), \operatorname{linspace}(K^*))) = \operatorname{lin}(K)\operatorname{lin}(K^*)$ giving equality. Substitute into the dimension formula to obtain the result. $\square$

This next fact lets us compute the orthogonal complement in Proposition 12.

Proposition 14. *If V is a finite-dimensional real Hilbert space and if $\{x, s, t, y\}$ is a subset of V, then $\langle s \otimes x, t \otimes y\rangle = 0$ if and only if $\langle x, y\rangle = 0$ or $\langle t, s\rangle = 0$.*

Proof. Let $L \coloneqq (s \otimes x) \circ (y \otimes t)$ so that $\langle s \otimes x, t \otimes y \rangle \coloneqq \operatorname{trace}(L)$. Note that $L(v) = \langle t, v \rangle \langle x, y \rangle s = \langle x, y \rangle (s \otimes t)(v)$, and so L has at most one nonzero eigenvalue $\lambda = \langle x, y \rangle \langle s, t \rangle$. The equivalence follows from $\operatorname{trace}(L) = \lambda$. □

Proposition 15. *If K is a closed convex cone in a finite-dimensional real Hilbert space V and if $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$, then $\operatorname{linspace}(\mathcal{T})^\perp$ is equal to*

$$\begin{aligned} &\operatorname{span}\left((K^*)^\perp \otimes \operatorname{linspace}(K)\right) \\ &\ominus \operatorname{span}\left(\operatorname{linspace}(K^*)^\perp \otimes \operatorname{linspace}(K)^\perp\right) \\ &\ominus \operatorname{span}\left(\operatorname{linspace}(K^*) \otimes K^\perp\right). \end{aligned} \tag{2.1}$$

Proof. Proposition 14 shows that the spans in (2.1) are mutually orthogonal to each other and to $\operatorname{linspace}(\mathcal{T})$. If the dimensions of (2.1) and $\operatorname{lin}(\mathcal{T})$ sum to $\dim(\mathcal{B}(V))$, then (2.1) must be $\operatorname{linspace}(\mathcal{T})^\perp$. To justify that assertion, note that the dimension of (2.1) is equal to

$$\operatorname{codim}(K^*) \operatorname{lin}(K) + \dim(K) \dim(K^*) + \operatorname{lin}(K^*) \operatorname{codim}(K).$$

Using Proposition 13, add $\operatorname{lin}(\mathcal{T})$ to that expression, and then eliminate the two codimension terms; for example,

$$\operatorname{codim}(K^*) \operatorname{lin}(K) + \operatorname{lin}(K) \dim(K^*) = \dim(V) \operatorname{lin}(K).$$

Further simplification yields $\dim(V)^2 = \dim(\mathcal{B}(V))$. □

Corollary 2. *If K is a closed convex cone in a finite-dimensional real Hilbert space and if $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$, then*

$$\operatorname{proj}\left(\operatorname{linspace}(\mathcal{T})^\perp, \mathcal{T}\right) \subseteq \operatorname{span}\left(\operatorname{linspace}(K^*)^\perp \otimes \operatorname{linspace}(K)^\perp\right).$$

Proof. Using Proposition 14, it is easy to see that $\operatorname{span}(\mathcal{T})$ is orthogonal to two of the three subspaces that constitute $\operatorname{linspace}(\mathcal{T})^{\perp}$ in Proposition 15. Therefore the projection of $\mathcal{T}$ onto $\operatorname{linspace}(\mathcal{T})^{\perp}$ lies entirely in the third subspace. $\square$

Corollary 2 shows that without loss of generality, the projection of $\mathcal{T}$ onto $\operatorname{linspace}(\mathcal{T})^{\perp}$ is onto $\operatorname{span}\left(\operatorname{linspace}(K^*)^{\perp} \otimes \operatorname{linspace}(K)^{\perp}\right)$ instead. This allows us to bring the following proposition and its corollary to bear, providing a description of the generators that appear when Proposition 10 is applied to $\mathcal{T}$.

Proposition 16. *Let V be a real Hilbert space with basis $\mathbf{v} := \{v_1, v_2, \ldots, v_n\}$. If W_1 and W_2 are subspaces of V, then the linear operator*

$$\Pi(v_i \otimes v_j) := \operatorname{proj}(W_1, v_i) \otimes \operatorname{proj}(W_2, v_j) \tag{2.2}$$

defines the orthogonal projection of $\mathcal{B}(V)$ onto $\operatorname{span}(W_1 \otimes W_2)$.

Proof. The existence and uniqueness of such a projection is guaranteed, because $\operatorname{span}(W_1 \otimes W_2)$ is a subspace of $\operatorname{span}(V \otimes V) = \mathcal{B}(V)$. A dimension argument shows that $\mathbf{v} \otimes \mathbf{v}$ is a basis for $\mathcal{B}(V)$, so (2.2) is well-defined, and clearly linear.

Observe from (2.2) that $\Pi(\mathcal{B}(V)) \subseteq \operatorname{span}(W_1 \otimes W_2)$ and that $\Pi(\phi) = \phi$ for any $\phi \in \operatorname{span}(W_1 \otimes W_2)$. If instead one takes any $\phi \in \operatorname{span}(W_1 \otimes W_2)^{\perp}$, then $\phi \in \operatorname{span}\left(W_1^{\perp} \otimes V\right) + \operatorname{span}\left(V \otimes W_2^{\perp}\right)$ and $\Pi(\phi) = 0$ by Proposition 14. The only such operator is the orthogonal projection onto $\operatorname{span}(W_1 \otimes W_2)$. $\square$

Corollary 3. *Let V be a finite-dimensional real Hilbert space and let W_1 and W_2 be subspaces of V. If $P \subseteq V \times V$, then*

$$\operatorname{proj}\left(\operatorname{span}(W_1 \otimes W_2), \operatorname{cone}(\{s \otimes x \mid (x, s) \in P\})\right)$$

is equal to

$$\operatorname{cone}\left(\left\{\operatorname{proj}\left(W_1, s\right) \otimes \operatorname{proj}\left(W_2, x\right) \mid (x, s) \in P\right\}\right).$$

Proof. Use Proposition 10 and then Proposition 16. $\square$

It will be be convenient to work with bounded generating sets of cones like $\operatorname{cone}\left(\left\{s \otimes x \mid (x, s) \in P\right\}\right)$ from Corollary 3. We construct such a set explicitly.

Proposition 17. *Let V be a finite-dimensional real Hilbert space. If $P \subseteq V \times V$ is nonempty and such that $(x, s) \in P$ implies that both $(\lambda x, s) \in P$ and $(x, \lambda s) \in P$ for all $\lambda \geq 0$, then*

$$\operatorname{cone}\left(\left\{s \otimes x \mid (x, s) \in P\right\}\right) = \operatorname{cone}\left(\left\{s \otimes x \mid (x, s) \in P, \|x\| = \|s\| = 1\right\}\right).$$

Proof. One set is clearly a subset of the other, so suppose that we are given a $\phi \in \operatorname{cone}\left(\left\{s \otimes x \mid (x, s) \in P\right\}\right)$. Pull out nonnegative factors so that

$$\phi = \sum_{i=1}^{m} \alpha_i \left(s_i \otimes x_i\right) = \sum_{i=1}^{m} \alpha_i \left\|s_i\right\| \left\|x_i\right\| \left(\frac{s_i}{\left\|s_i\right\|} \otimes \frac{x_i}{\left\|x_i\right\|}\right).$$

The latter belongs to $\operatorname{cone}\left(\left\{s \otimes x \mid (x, s) \in P, \|x\| = \|s\| = 1\right\}\right)$. $\square$

The set P in Proposition 17 simply generalizes the two sets $K \times K^*$ and $C(K)$.

Lemma 1. *If K is a closed convex cone in a finite-dimensional real Hilbert space and if $\mathcal{T} = \operatorname{cone}\left(K^* \otimes K\right)$, then $\mathcal{T}$ is closed.*

Proof. It suffices by Proposition 9 to show that $\operatorname{proj}\left(\operatorname{linspace}\left(\mathcal{T}\right)^{\perp}, \mathcal{T}\right)$ is closed. We will construct a generating set G of that projection such that $\operatorname{cone}(G)$ is closed. First, use Corollary 2 to show that $\operatorname{proj}\left(\operatorname{linspace}\left(\mathcal{T}\right)^{\perp}, \mathcal{T}\right)$ is equal to

$$\operatorname{proj}\left(\operatorname{span}\left(\operatorname{linspace}\left(K^*\right)^{\perp} \otimes \operatorname{linspace}\left(K\right)^{\perp}\right), \mathcal{T}\right). \tag{2.3}$$

Then, from Corollary 3 and the fact that $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$, the set (2.3) is

$$\operatorname{cone}\left(\operatorname{proj}\left(\operatorname{linspace}(K^*)^\perp, K^*\right) \otimes \operatorname{proj}\left(\operatorname{linspace}(K)^\perp, K\right)\right). \tag{2.4}$$

If either projection in (2.4) is $\{0\}$, then the entire cone is the closed set $\{0\}$ and the proof is complete. We can therefore assume that both projections contain nonzero elements. And since they are both closed convex cones by Proposition 9, they contain elements of unit norm. Thus we have two nonempty compact sets,

$$J_1 := \left\{x \in \operatorname{proj}\left(\operatorname{linspace}(K)^\perp, K\right) \;\middle|\; \|x\| = 1\right\}$$
$$J_2 := \left\{s \in \operatorname{proj}\left(\operatorname{linspace}(K^*)^\perp, K^*\right) \;\middle|\; \|s\| = 1\right\}.$$

Use Proposition 17 to see that $\operatorname{cone}(J_2 \otimes J_1)$ is equal to (2.4), and let $G := J_2 \otimes J_1 \neq \emptyset$. The set G is also compact, since it is the image of the continuous mapping $(x, s) \mapsto s \otimes x$ on the compact set $J_1 \times J_2$. It follows that $\operatorname{conv}(G)$ is compact [1]. For $\operatorname{cone}(G)$ to be closed, it suffices [31, 38] to show that $0 \notin \operatorname{conv}(G)$. Suppose on the contrary that

$$0 = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m, \text{ where } g_i \in G,\ \lambda_i > 0, \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

This can be rearranged to $-\lambda_1 g_1 = \lambda_2 g_2 + \cdots + \lambda_m g_m \in \mathcal{T}$, contradicting the fact that $g_1 \in \operatorname{linspace}(\mathcal{T})^\perp$ unless $g_1 = 0$. The elements of G were constructed to have nonzero norm, so that cannot happen. □

Theorem 1. *If $K = \operatorname{cone}(G_1)$ is closed in a finite-dimensional real Hilbert space and if $K^* = \operatorname{cone}(G_2)$, then $\pi(K)^* = \operatorname{cone}(G_2 \otimes G_1)$.*

Proof. Expand the elements of K and K^* in terms of G_1 and G_2 to verify that $\operatorname{cone}(G_2 \otimes G_1) = \operatorname{cone}(K^* \otimes K)$. Since K is closed, we have $(K^*)^* = K$, and thus $L \in \pi(K)$ if and only if $\langle L(x), s\rangle \geq 0$ for all $(x, s) \in K \times K^*$. The identity $\langle L(x), s\rangle = \langle L, s \otimes x\rangle$ is a property of the trace, so it follows that

$$\langle L, s \otimes x\rangle \geq 0 \text{ for all } (x, s) \in K \times K^* \iff L \in \operatorname{cone}(G_2 \otimes G_1)^*.$$

Therefore $\pi(K) = \operatorname{cone}(G_2 \otimes G_1)^*$, and Lemma 1 shows that $\operatorname{cone}(G_2 \otimes G_1)$ is closed. Take duals on both sides. $\square$

Example 7. If $K = \{0\}$ in V, then there are no nonzero generators of K, so $\pi(K)^* = \{0\}$, and thus $\pi(K) = \mathcal{B}(V)$. Likewise if $K = V$ and $K^* = \{0\}$.

Example 8. If $K = \operatorname{cone}(\{e_1, \pm e_2\})$ is the right half-space in $V = \mathbb{R}^2$, then $K^* = \operatorname{cone}(\{e_1\})$ and Theorem 1 gives

$$\pi(K)^* = \operatorname{cone}\left(\left\{e_1 e_1^T, \pm e_1 e_2^T\right\}\right); \ \pi(K) = \operatorname{cone}\left(\left\{e_1 e_1^T, \pm e_2 e_1^T, \pm e_2 e_2^T\right\}\right).$$

This result is verified using Proposition 7.

Using extreme directions, Tam proved the following for proper cones [38]. We extend the result to all closed convex cones.

Theorem 2. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then $\pi(K)$ is polyhedral if and only if K is polyhedral.*

Proof. Necessity follows directly from Theorem 1 and Proposition 4, so suppose that $\pi(K)$ is polyhedral. Then $\pi(K)^*$ is polyhedral as well by Proposition 4. From Theorem 1 we have $\pi(K)^* = \operatorname{cone}(G_2 \otimes G_1)$ for generating sets G_1 and G_2 of K and K^* respectively. Since $\pi(K)^*$ is polyhedral, it is generated by a finite set P. Each $p \in P$ belongs to $\pi(K)^*$, so by Carathéodory's theorem, it is a finite conic combination of elements of $G_2 \otimes G_1$. Collect these elements—a finite number for each p in the finite set P—to construct a finite subset $F_1 \times F_2 \subseteq G_1 \times G_2$ such that $\pi(K)^* = \operatorname{cone}(F_2 \otimes F_1)$. We show that $\operatorname{cone}(F_2) = K^*$ and it follows that both K^* and K are polyhedral.

If $K = \operatorname{linspace}(K)$, then K is polyhedral, so assume otherwise. Choose a nonzero $x \in K \cap \operatorname{linspace}(K)^{\perp}$ and any $s \in K^*$. Then $s \otimes x \in \pi(K)^*$, so

$$s \otimes x = \sum_{i=1}^{m} \alpha_i (s_i \otimes x_i) \text{ where } \alpha_i \geq 0 \text{ and } (x_i, s_i) \in F_1 \times F_2.$$

Since $x \notin \operatorname{linspace}(K)$, there exists a $t \in K^*$ such that $\langle x, t \rangle > 0$ by Corollary 1. Apply $s \otimes x$ to t to obtain $s = \frac{1}{\langle x,t \rangle} \sum_{i=1}^{m} \alpha_i \langle x_i, t \rangle s_i \in \operatorname{cone}(F_2)$. $\square$

Proposition 12 and Proposition 13 now have intepretations in terms of $\pi(K)$.

Corollary 4. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then* $\operatorname{linspace}(\pi(K)^*) = U_1 + U_2$ *where*

$$U_1 \coloneqq \operatorname{span}(K^* \otimes \operatorname{linspace}(K)); \quad U_2 \coloneqq \operatorname{span}(\operatorname{linspace}(K^*) \otimes K).$$

Proof. Combine Proposition 12 with Theorem 1. $\square$

Corollary 5. *Let K be a closed convex cone in a finite-dimensional real Hilbert space V. If $n = \dim(V)$, $m = \dim(K)$, and $\ell = \operatorname{lin}(K)$, then $\operatorname{lin}(\pi(K)^*) = \ell(m - \ell) + m(n - m)$.*

Proof. One form of $\operatorname{lin}(\pi(K)^*)$ is given by Theorem 1 and Proposition 13. Use Proposition 2 to achieve the desired form. $\square$

Lemma 2. *Let K be a closed convex cone in a finite-dimensional real Hilbert space V, and let $n = \dim(V)$, $m = \dim(K)$, and $\ell = \operatorname{lin}(K)$. Then,*

$$\dim(\pi(K)) = n^2 - \ell(m-\ell) - m(n-m).$$

Proof. Apply Corollary 5 to $\operatorname{codim}(\pi(K)) = \operatorname{lin}(\pi(K)^*)$. $\square$

Example 9. If $K = \{0\}$ in V, then $m = \ell = 0$, and $\dim(\pi(K)) = n^2$ which agrees with the obvious fact that $\pi(K) = \mathcal{B}(V)$.

Example 10. If K is proper, then in Lemma 2, we have $m = n$ and $\ell = 0$. Thus $\dim(\pi(K)) = n^2$ and $\pi(K)$ is solid.

Example 11. Example 8 has $n = m = 2$ and $\ell = 1$ giving $\dim(\pi(K)) = 3$.

Lemma 3. *If K is a closed convex cone in a finite-dimensional real Hilbert space V, then*

$$\operatorname{lin}(\pi(K)) = \dim(V)^2 - \dim(K)\dim(K^*).$$

Proof. By Proposition 2 we have $\operatorname{lin}(\pi(K)) = \dim(V)^2 - \dim(\pi(K)^*)$, and from Theorem 1 it follows that $\dim(\pi(K)^*) = \dim(K)\dim(K^*)$. $\square$

Example 12. If $K = \{0\}$ in V, then $\dim(K) = 0$, and $\operatorname{lin}(\pi(K)) = \dim(V)^2$ in agreement with the fact that $\pi(K) = \mathcal{B}(V)$.

Example 13. If K is proper, then $\dim(K) = \dim(K^*) = \dim(V)$. Lemma 3 gives $\operatorname{lin}(\pi(K)) = 0$, showing that $\pi(K)$ is pointed.

Example 14. In Example 8, we have $\operatorname{lin}(\pi(K)) = 4 - 2 \cdot 1 = 2$.

These corollaries and examples reaffirm that if K is proper, then $\pi(K)$ is proper [34]. Lemma 3 allows us to prove the converse.

Theorem 3. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then $\pi(K)$ is proper if and only if K is proper.*

When K is polyhedral, Theorem 1 allows us to compute a generating set of $\pi(K)$. Algorithms to compute the dual generators of a polyhedral cone are known, and the inverse operations $\operatorname{vec}()$ and $\operatorname{mat}()$ are isometries.

Algorithm 1 Compute generators of $\pi(K)$

Input: A closed convex cone K

Output: A generating set of $\pi(K)$

function PI(K)

  $G_1 \leftarrow$ a finite set of generators for K

  $G_2 \leftarrow \operatorname{dual}(G_1)$ ▷ a finite set of generators for K^*

  $G \leftarrow G_2 \otimes G_1$

  return $\operatorname{mat}(\operatorname{dual}(\operatorname{vec}(G)))$

end function

2.2 **Z**-operators

We now move on to the **Z**-operators introduced in Definition 15. Every **Z**-operator is the negation of some *cross-positive* operator—the class introduced by Schneider and Vidyasagar [35]. As before, we begin by pointing out that the set of all **Z**-operators on K forms a closed convex cone. Verification of the three criteria is straightforward, but in this case, none of them depend on properties of K.

Proposition 18. *$\mathbf{Z}(K)$ is a closed convex cone for any set K.*

If the ambient space is nontrivial, then $\mathbf{Z}(K)$ contains the nontrivial subspace $\mathbf{LL}(K)$ and is never proper in contrast with Theorem 3.

Proposition 19. *If $K = \operatorname{cone}(G_1)$ is closed in a finite-dimensional real Hilbert space and if $K^* = \operatorname{cone}(G_2)$, then $L \in \mathbf{Z}(K)$ if and only if*

$$\langle L(x), s\rangle \leq 0 \text{ for all } (x, s) \in C(K) \cap (G_1 \times G_2). \tag{2.5}$$

Proof. Clearly, if $L \in \mathbf{Z}(K)$, then L satisfies (2.5). So suppose that L satisfies (2.5) and let $(x, s) \in C(K)$. Since G_1 generates K and G_2 generates K^*, we can write $x = \sum_{i=1}^{\ell} \alpha_i x_i$ and $s = \sum_{j=1}^{m} \gamma_j s_j$. By expanding $\langle x, s\rangle = 0$ and noting that $\langle x_i, s_j\rangle \geq 0$, we see that each $(x_i, s_j) \in C(K)$. Linearity gives $\langle L(x), s\rangle \leq 0$. □

As with $\pi(K)^*$, we will eventually want to find a generating set of $\mathbf{Z}(K)^*$ and use that to prove some results about $\mathbf{Z}(K)$. Recall the cone of Tam's generators $\mathcal{T} \coloneqq \operatorname{cone}(K^* \otimes K)$ from Section 2.1. There we showed that $\mathcal{T} = \pi(K)^*$, and a similar set will generate $\mathbf{Z}(K)^*$. When K is a closed convex cone, we will now define

$$\mathcal{T}_C \coloneqq \operatorname{cone}(\{s \otimes x \mid (x, s) \in C(K)\}).$$

Proposition 20. *If K is a closed convex cone in a finite-dimensional real Hilbert space V and if $\mathcal{T}_C = \operatorname{cone}(\{s \otimes x \mid (x, s) \in C(K)\})$, then we can express $\mathcal{T}_C$ as the intersection of $\mathcal{T} \coloneqq \operatorname{cone}(K^* \otimes K)$ and a subspace, namely $\mathcal{T}_C = \mathcal{T} \cap \mathrm{id}_V^{\perp}$.*

Proof. Clearly $\mathcal{T}_C \subseteq \mathcal{T}$, and if $L = \sum s_i \otimes x_i \in \mathcal{T}_C$, then $\langle L, \mathrm{id}_V\rangle = 0$ because each $s_i \otimes x_i$ has trace zero and the trace is linear. So $\mathcal{T}_C \subseteq \mathrm{id}_V^{\perp}$.

On the other hand, if $L = \sum s_i \otimes x_i \in \mathcal{T}$ with $\langle L, \mathrm{id}_V \rangle = \mathrm{trace}\,(L) = 0$, then the linearity of the trace and the fact that $\mathrm{trace}\,(s_i \otimes x_i) = \langle s_i, x_i \rangle \geq 0$ imply that each $\langle s_i, x_i \rangle = 0$. Thus $L \in \mathcal{T}_C$. ☐

Lemma 4. *If K is a closed convex cone in a finite-dimensional real Hilbert space and if $\mathcal{T}_C = \mathrm{cone}\,(\{s \otimes x \mid (x, s) \in C\,(K)\})$, then $\mathcal{T}_C$ is closed.*

Proof. Use Proposition 20 to write $\mathcal{T}_C$ as the intersection of a cone and a subspace. Lemma 1 says that the cone is closed. ☐

Theorem 4. *If $K = \mathrm{cone}\,(G_1)$ is closed in a finite-dimensional real Hilbert space and if $K^* = \mathrm{cone}\,(G_2)$, then $\mathbf{Z}\,(K)^* = \mathrm{cone}\,(G)$ where G is defined to be $G :=$ $\{-s \otimes x \mid (x, s) \in C\,(K) \cap (G_1 \times G_2)\}$. In particular, if K is a closed convex cone, then $\mathbf{Z}\,(K)^* = \mathrm{cone}\,(\{s \otimes x \mid (x, s) \in C\,(K)\})$.*

Proof. Follow Theorem 1. Let $\mathcal{T}_C = \mathrm{cone}\,(\{s \otimes x \mid (x, s) \in C\,(K)\})$; we will show that $\mathrm{cone}\,(G) = -\mathcal{T}_C$ which by Lemma 4 is closed. Obviously, $\mathrm{cone}\,(G) \subseteq -\mathcal{T}_C$, so let $h \in -\mathcal{T}_C$ be given with $h = \sum_{i=1}^{\ell} -s_i \otimes x_i$ where each $(x_i, s_i) \in C\,(K)$. Take any term $-t \otimes y$ in this sum, and write t and y in terms of G_2 and G_1,

$$-t \otimes y = -\left(\sum_{k=1}^{n} \gamma_k h_k\right) \otimes \left(\sum_{j=1}^{m} \alpha_j g_j\right) = \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_j \gamma_k \left(-h_k \otimes g_j\right).$$

Expand $\langle y, t \rangle = 0$ and note that each $\langle g_j, h_k \rangle \geq 0$ implying $(g_j, h_k) \in C\,(K)$. Thus each $-t \otimes y \in \mathrm{cone}\,(G)$, and so $h \in \mathrm{cone}\,(G)$. Conclude that $\mathrm{cone}\,(G) = -\mathcal{T}_C$. By definition, $L \in \mathbf{Z}\,(K)$ if and only if $\langle -L\,(x), s \rangle \geq 0$ for all $(x, s) \in C\,(K)$. By a property of the trace we have $\langle -L\,(x), s \rangle = \langle L, -s \otimes x \rangle$, so it follows that

$$L \in \mathbf{Z}\,(K) \iff \langle L, -s \otimes x \rangle \geq 0 \text{ for all } (x, s) \in C\,(K) \iff L \in \mathrm{cone}\,(G)^*.$$

Take duals on both sides of $\mathbf{Z}(K) = \operatorname{cone}(G)^*$. $\square$

There is no corresponding result for proper cones, so we include the classical case along with some improper cones as examples of Theorem 4.

Example 15. If $K = \mathbb{R}_+^n$ in $V = \mathbb{R}^n$, then $C(K) = \{(e_i, e_j) \mid i \neq j\}$. Form $G := \left\{-e_j e_i^T \,\middle|\, i \neq j\right\}$ to find that $\mathbf{Z}(K)^* = \operatorname{cone}(G)$ is the set of matrices whose diagonal entries are zero and whose off-diagonal entries are nonpositive. Its dual is the cone of $\mathbf{Z}$-matrices.

Example 16. If $K = \{0\}$ or $K = V$, then $\mathbf{Z}(K)^* = \{0\}$ and $\mathbf{Z}(K) = \mathcal{B}(V)$.

Example 17. If K is the half-space from Example 8, then Theorem 4 gives $\mathbf{Z}(K)^* = \mathbf{Z}(K)^\perp = \operatorname{span}\left(\left\{e_1 e_2^T\right\}\right)$. This result is verified by Proposition 19.

Now that we know that $\mathbf{Z}(K)^* = -\mathcal{T}_C$, we can find the dimension of $\mathbf{Z}(K)$.

Proposition 21. *Let K be a closed convex cone in a finite-dimensional real Hilbert space V. If $\mathcal{T} = \operatorname{cone}(K^* \otimes K)$ and if $\mathcal{T}_C = \operatorname{cone}(\{s \otimes x \mid (x, s) \in C(K)\})$, then $\operatorname{linspace}(\mathcal{T}_C) = \operatorname{linspace}(\mathcal{T})$.*

Proof. Use Proposition 8 to $\mathcal{T}$ and Proposition 20 to show that $\operatorname{linspace}(\mathcal{T}_C) = \operatorname{linspace}(\mathcal{T}) \cap \operatorname{id}_V^\perp$. If $L = \sum s_i \otimes x_i \in \mathcal{T}$, then $\langle L, \operatorname{id}_V \rangle = \sum \langle s_i, x_i \rangle \geq 0$ and $\pm L \in \mathcal{T}$ would imply that $\langle L, \operatorname{id}_V \rangle = 0$. Thus $\operatorname{linspace}(\mathcal{T}) \subseteq \operatorname{id}_V^\perp$. $\square$

Theorem 5. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then $\dim(\mathbf{Z}(K)) = \dim(\pi(K))$.*

Proof. Theorem 1, Proposition 21, and Theorem 4 show that $\operatorname{linspace}(\mathbf{Z}(K)^*) = \operatorname{linspace}(\pi(K)^*)$. Proposition 2 gives $\operatorname{codim}(\mathbf{Z}(K)) = \operatorname{codim}(\pi(K))$. $\square$

The trivial cone, half-space, full space, and nonnegative orthant all corroborate Theorem 5. Our next result is an analogue of Theorem 2 and its proof is similar.

Theorem 6. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then $\mathbf{Z}(K)$ is polyhedral if and only if K is polyhedral.*

Proof. Refer to the proof of Theorem 2. Necessity follows immediately from Theorem 4 and Proposition 4, so suppose that $\mathbf{Z}(K)$ is polyhedral. Without loss of generality, $\mathbf{Z}(K)^{*} = \operatorname{cone}(-F_2 \otimes F_1)$ for finite sets $F_1 \subseteq K$ and $F_2 \subseteq K^{*}$. If $\dim(K^{*}) = 1$, or if K^{*} is a (half) subspace, then K is polyhedral. Fenchel's Theorem 13 lets us assume that K^{*} is generated by its relative boundary rays [11].

Let s be any nonzero relative boundary ray of K^{*}. Define the closed convex cone $J \coloneqq K^{*} + \operatorname{linspace}(K)$ which by Proposition 2 equals $K^{*} \oplus \operatorname{span}(K^{*})^{\perp}$. Then s lies on the boundary of J, and by Proposition 5, there exists a nonzero $x \in J^{*}$ such that $\langle x, s \rangle = 0$. Proposition 3 gives $J^{*} = K \cap \operatorname{linspace}(K)^{\perp}$, so this pair satisfies $(x, s) \in C(K)$ and thus $-s \otimes x \in \mathbf{Z}(K)^{*}$. The argument used in the proof of Theorem 2 can be repeated to show that $s \in \operatorname{cone}(F_2)$. $\square$

This result is confirmed by the polyhedral cones we have examined—all have polyhedral cones of $\mathbf{Z}$-operators.

Corollary 6. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then $\mathbf{Z}(K)$ is polyhedral if and only if $\pi(K)$ is polyhedral.*

There are no simple characterizations of $\mathbf{Z}(K)$ for nonpolyhedral K. One sees an example in the work of Stern and Wolkowicz [37] who characterize the $\mathbf{Z}$-

operators on the Lorentz "ice cream" cone. We close this section with an algorithm, based on Theorem 4, to compute $\mathbf{Z}(K)$ for polyhedral K.

Algorithm 2 Compute generators of $\mathbf{Z}(K)$

Input: A closed convex cone K

Output: A generating set of $\mathbf{Z}(K)$

```
function Z(K)
    G_1 ← a finite set of generators for K
    G_2 ← dual(G_1)                          ▷ a finite set of generators for K*
    G ← {−s ⊗ x | x ∈ G_1, s ∈ G_2, ⟨x, s⟩ = 0}
    return mat(dual(vec(G)))
end function
```

2.3 Lyapunov-like operators

Using what we now know about $\mathbf{Z}$-operators, we can extend the theory of Lyapunov-like operators to all closed convex cones.

Proposition 22. *If $K = \operatorname{cone}(G_1)$ is closed in a finite-dimensional real Hilbert space and if $K^{*} = \operatorname{cone}(G_2)$, then $L \in \mathbf{LL}(K)$ if and only if*

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K) \cap (G_1 \times G_2). \tag{2.6}$$

Proof. Apply Proposition 19 to $\pm L$. □

Proposition 22 and a generating set for K will often allow us to describe $\mathbf{LL}(K)$ and determine $\beta(K)$. We illustrate this with an example.

Example 18. Let K be the xy-plane in $V = \mathbb{R}^3$. Then K^* is the z-axis, and they have the respective generating sets

$$K = \operatorname{cone}\left(\{\pm e_1, \pm e_2\}\right); \quad K^* = \operatorname{cone}\left(\{\pm e_3\}\right).$$

Using Proposition 22, one can verify that neither E_{31} nor E_{32} is Lyapunov-like on K but that the remaining seven E_{ij} are. Thus, $\beta\left(K\right) = \dim\left(\mathbf{LL}\left(K\right)\right) = 7$.

Finding a tight upper bound for the Lyapunov rank of a proper cone is an open problem that we address in Chapter 4. The following example shows that, in general, there can be no similar bound for closed convex cones.

Example 19. Suppose $K = V = \mathbb{R}^n$. Then $K^* = \{0\}$ and $C\left(K\right) = K \times \{0\}$, so every $L \in \mathcal{B}\left(V\right)$ is Lyapunov-like on K and $\dim\left(\mathcal{B}\left(V\right)\right) = n^2$.

The following two results generalize known results [33] for proper cones.

Proposition 23. *If K is a closed convex cone in a finite-dimensional real Hilbert space, then the Lyapunov ranks $\beta\left(K\right)$ and $\beta\left(K^*\right)$ are equal.*

Proof. It follows from Definition 17 that $L \in \mathbf{LL}\left(K\right)$ if and only if $L^* \in \mathbf{LL}\left(K^*\right)$. The map $L \mapsto L^*$ is bijective, so $\dim\left(\mathbf{LL}\left(K\right)\right) = \dim\left(\mathbf{LL}\left(K^*\right)\right)$. □

Proposition 24. *Let V and W be two finite-dimensional real Hilbert spaces. If K is a closed convex cone in V and if $A \in \mathcal{B}\left(V, W\right)$ is invertible, then $\beta\left(K\right) = \beta\left(A\left(K\right)\right)$.*

Proof. We first observe that $A\left(K\right)^* = \left(A^*\right)^{-1}\left(K^*\right)$. Then it is evident that $L \in \mathbf{LL}\left(K\right) \iff ALA^{-1} \in \mathbf{LL}\left(A\left(K\right)\right)$. The result follows from the fact that $L \mapsto ALA^{-1}$ is an invertible linear operator from $\mathcal{B}\left(V\right)$ to $\mathcal{B}\left(W\right)$. □

Recall the linear complementarity problem of Example 5. There it was implied that $\beta\left(\mathbb{R}^n_+\right) = n$ which led to a possible means of solution to the problem via the system (1.2). That possibility exists whenever the complementarity set can be decomposed via a large number of Lyapunov-like operators.

Definition 21. A closed convex cone K in a finite-dimensional real Hilbert space V is *perfect* if there exist $\dim(V)$ linearly-independent $L_i \in \mathbf{LL}(K)$ such that

$$C(K) = \bigcap_{i=1}^{\dim(V)} \{(x,s) \in K \times K^* \mid \langle L_i(x), s\rangle = 0\}.$$

If K is perfect, the equation $\langle x, s\rangle = 0$ can be rewritten as a system of $\dim(V)$ equations, $\langle L_i(x), s\rangle = 0$, for $1 \leq i \leq \dim(V)$. As the name "perfect" implies, this is desirable because it may lead to a soluble square system.

Example 20. In the linear complementarity problem of Example 5, the cone $K = \mathbb{R}^n_+$ is perfect. Let $L_i = E_{ii}$ for $1 \leq i \leq n$. Then,

$$\begin{aligned}
C(K) &= \{(x,s) \mid x \geq 0, s \geq 0, \langle x, s\rangle = 0\} \\
&= \bigcap_{i=1}^{n} \{(x,s) \mid x \geq 0, s \geq 0, x_i s_i = 0\} \\
&= \bigcap_{i=1}^{n} \{(x,s) \mid x \geq 0, s \geq 0, \langle L_i(x), s\rangle = 0\}.
\end{aligned}$$

According to Definition 21, the cone $\mathbb{R}^n_+$ is perfect.

We can characterize perfect cones by their Lyapunov ranks. This settles an open question [19], and provides two conditions by which a cone can be found perfect.

Theorem 7. *If K is a closed convex cone in a finite-dimensional real Hilbert space V, then the following are equivalent:*

(i) The identity operator id_V *on* V *can be written as a linear combination of* $\dim(V)$ *elements of* $\mathbf{LL}(K)$*.*

(ii) K *is perfect.*

(iii) $\beta(K) \geq \dim(V)$*.*

Proof. Gowda and Tao [19] proved that item (i) implies the others. It remains to show that if $\beta(K) \geq \dim(V)$, then id_V is a linear combination of $\dim(V)$ Lyapunov-like operators on K. But id_V is Lyapunov-like on K, so extend the set $\{\mathrm{id}_V\}$ to a basis $\left\{\mathrm{id}_V, L_2, \ldots, L_{\beta(K)}\right\}$ of $\mathbf{LL}(K)$. Then $\mathrm{id}_V = \mathrm{id}_V + 0L_2 + \cdots + 0L_{\dim(V)}$. □

Rudolf et al. gave a codimension formula for the Lyapunov rank [33], and an analogous formula holds for all closed convex cones.

Theorem 8. *If* $K = \mathrm{cone}(G_1)$ *is closed in a finite-dimensional real Hilbert space and if* $K^* = \mathrm{cone}(G_2)$*, then*

$$\beta(K) = \mathrm{codim}\left(\mathrm{span}\left(\{s \otimes x \mid (x, s) \in C(K) \cap (G_1 \times G_2)\}\right)\right). \tag{2.7}$$

Proof. We know that $\mathbf{LL}(K) = \mathrm{linspace}(\mathbf{Z}(K))$, so from Proposition 2 we have $\mathbf{LL}(K)^{\perp} = \mathrm{span}(\mathbf{Z}(K)^*)$. Note the generators of $\mathbf{Z}(K)^*$ from Theorem 4 and compute $\dim(\mathbf{LL}(K)) = \mathrm{codim}\left(\mathbf{LL}(K)^{\perp}\right)$. □

As an application, we compute the Lyapunov rank of a vector subspace.

Proposition 25. *Suppose* V *is a* n*-dimensional real Hilbert space. If* K *is an* m*-dimensional subspace of* V*, then* $\beta(K) = n^2 - m(n - m)$*.*

Proof. Using Proposition 24, we can assume that $V = \mathbb{R}^n$ with the standard basis and that $K = \mathbb{R}^m$. Now $G_1 \coloneqq \{\pm e_i\}_{i=1}^m$ and $G_2 \coloneqq \{\pm e_i\}_{i=m+1}^n$ generate K and $K^* = \mathbb{R}^{n-m}$ respectively. Thus,

$$C(K) \cap (G_1 \times G_2) = \{(\pm e_i, \pm e_j) \mid i \leq m;\ m+1 \leq j \leq n\}.$$

As $\operatorname{span}(\{s \otimes x \mid (x, s) \in C(K) \cap (G_1 \times G_2)\})$ reduces to the span of some E_{ji}, it follows from (2.7) that $\beta(K) = n^2 - m(n-m)$. □

Note that this agrees with Example 18 where $n = 3$, $m = 2$, and $\beta(K) = 7$.

Proposition 26. *Let V be an n-dimensional real Hilbert space. If $K = \operatorname{cone}(\{v\})$ for some nonzero $v \in V$, then $\beta(K) = n^2 - n + 1$.*

Proof. Without loss of generality, we can take K to be $\operatorname{cone}(\{e_1\})$ and V to be $\mathbb{R}^n$. Then K^* is the right half-space containing e_1 in $\mathbb{R}^n$. It is obvious that $G_1 \coloneqq \{e_1\}$ generates K and $G_2 \coloneqq \{e_1\} \cup \{\pm e_j \mid j > 1\}$ generates K^*. Consider the pairs (e_1, e_2) through (e_1, e_n) in (2.7) to conclude that $\beta(K) = n^2 - (n-1)$. □

Corollary 7. *The Lyapunov rank of any ray, half-space, line, or hyperplane in an n-dimensional real Hilbert space is $n^2 - n + 1$.*

Proof. The half-space is dual to a single ray, and we can apply Proposition 26 to the set containing a single ray. The line/hyperplane are also duals, and their complementarity sets differ only in sign from those of the ray/half-space. □

Proposition 9 of Rudolf et al. [33] shows that Lyapunov rank is additive on a Cartesian product when its factors are proper cone-space pairs.

Proposition 27 (Rudolf et al.)**.** *If K and J are proper cones in a finite-dimensional real Hilbert space, then $\beta(K \times J) = \beta(K) + \beta(J)$.*

Surprisingly, this does not hold in general. If $K = \operatorname{cone}(\{e_1\})$ in $\mathbb{R}^n$, then informally, K can be written as the product $\operatorname{cone}(\{e_1\}) \times \{0\}$ where $\operatorname{cone}(\{e_1\})$ lives in $\mathbb{R}$ and $\{0\}$ lives in $\mathbb{R}^{n-1}$. Apply Proposition 27 to that product:

$$\beta(\operatorname{cone}(\{e_1\}) \times \{0\}) = \beta(\operatorname{cone}(\{e_1\})) + \beta(\{0\}).$$

Proposition 25 and Proposition 26 give $\beta(\{0\}) = (n-1)^2$ and $\beta(\operatorname{cone}(\{e_1\})) = 1$, respectively, so $\beta(K) = 1 + (n-1)^2$. Now apply Proposition 26 directly to K to obtain $\beta(K) = n^2 - n + 1$. These two results disagree when $n \geq 2$, and this example can be formalized to show that Proposition 27 is invalid for improper cones.

Chapter 3

Lyapunov rank and the exponential connection

In this chapter, we connect the Lyapunov-like operators on a closed convex cone to the Lie algebra of its automorphism group. The analogous result for proper cones was derived by Gowda and Tao [19], inspired by the earlier work of Schneider and Vidyasagar [35] on positive and **Z**-operators.

To simplify the notation, we first introduce the concept of a cone-space pair. We then find the Lyapunov rank of a closed convex cone in terms of a proper subcone. We present an algorithm to perform the computation, and then finally make the connection between a cone's Lyapunov-like operators and its automorphism group.

3.1 Cone-space pairs

Many operations that we have performed on cones so far have depended implicitly on the ambient space V. When K is a proper cone in V, there is no ambiguity—no smaller space contains K. In this chapter, if K lives in a proper subspace W of V, then we will need to (for example) take the dual of K within W. The usual notation does not allow this: the expression "K^*" is ambiguous when we may think of K as a subset of more than one ambient space, since "K^* in V" and "K^* in W" are two different sets. To avert that ambiguity, we make the following definition.

Definition 22. A *cone-space pair* (K, V) is a closed convex cone K paired with a finite-dimensional real Hilbert space V containing K.

We avoid the cumbersome pair notation with the following device.

Definition 23. If (K, V) is a cone-space pair and if W is a subspace of V, then we define a new cone-space pair $K_W \coloneqq (K \cap W, W)$. We extend this "operation" to cone-space pairs by $(K_W)_U = (K_U)_W = K_{UW} \coloneqq (K \cap U \cap W, U \cap W)$.

Note that $K_V = (K \cap V, V) = (K, V)$ whenever K is contained in V; this motivates an abuse of notation when we say "let K_V be a cone-space pair" to mean "let (K, V) be a cone-space pair."

The "space" in "cone-space pair" (that is, the subscript) is mainly a bookkeeping tool. Any operation defined on a closed convex cone K in a finite-dimensional real Hilbert space V can be defined on the cone-space pair K_V in an obvious way: think of K as a subset of V, perform the operation, and (if necessary) pair the result with the appropriate space. Here are a few examples.

Definition 24. The dual cone-space pair of K_V is another cone-space pair,

$$K_V^* \coloneqq (\{y \in V \mid \forall x \in K,\ \langle x, y\rangle \geq 0\}, V)\,.$$

We define $\operatorname{codim}(K_V)$ in terms of the orthogonal complement of K in V:

$$\operatorname{codim}(K_V) \coloneqq \dim(\{y \in V \mid \forall x \in K,\ \langle x, y\rangle = 0\})\,.$$

Proposition 28. *If K_V is a cone-space pair and if W is a subspace of V containing K, then $(K_W)^* = (K_V^*)_W$.*

We will freely perform operations on cone-space pairs that we defined on subsets of Hilbert spaces. There is no ambiguity if the space is treated as an annotation.

For example, any $\phi \in \mathcal{B}(V, W)$ acts on a cone-space pair by $\phi(K_V) = \phi(K)_W$. Subspaces are closed convex cones, but we will not belabor the notation. If W is a subspace of V, we write $W^\perp$ and not $W_V^\perp$ for its orthogonal complement in V.

Definition 25. Two cone-space pairs K_V and J_W are *isometric* if there exists an isometry $\phi : V \to W$ with $\phi(K) = J$.

One should beware of terms like "solid cone" that implicitly refer to an ambient space. Those definitions will now refer to the explicit space component of the pair. If K lives in a subspace W of the ambient space V, then the cone-space pair K_W is solid if $\operatorname{span}(K) = W$ and not if $\operatorname{span}(K) = V$.

3.2 The Lyapunov rank of an improper cone

The failure of the product formula in Proposition 27 motivates us to find a similar formula that works for all closed convex cones. In that counterexample, we informally wrote K as a product of two cones. The first factor was solid in $\operatorname{span}(K)$, and the second factor was trivial in $\operatorname{span}(K)^\perp$. Once we formalize that procedure, it will become a major tool used in the remainder of this chapter.

Proposition 29. *If K_V is a cone-space pair and if W is a subspace of V containing K, then V is isometric to $W \times W^\perp$ and $K_V \cong K_W \times \{0\}_{W^\perp}$.*

Proof. Suppose $\{f_i\}_{i=1}^m$ and $\{g_j\}_{j=m+1}^n$ are bases for W and $W^\perp$, respectively. Define the isometry $\phi : V \to W \times W^\perp$ by $\phi(f_i) = (f_i, 0)^T$ and $\phi(g_j) = (0, g_j)^T$. □

Proposition 29 and Proposition 24 show that we can find $\beta(K_V)$ by computing

$\beta\left(K_W \times \{0\}_{W^\perp}\right)$ instead. When K_V is non-solid, the latter is simpler.

Proposition 30. *If K_V is a cone-space pair and if $S = \operatorname{span}(K)$, then*

$$\pi\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \;\middle|\; A \in \pi\left(K_S\right) \right\},$$
$$\mathbf{Z}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \;\middle|\; A \in \mathbf{Z}\left(K_S\right) \right\}, \text{ and}$$
$$\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right) = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \;\middle|\; A \in \mathbf{LL}\left(K_S\right) \right\}$$

where $B \in \mathcal{B}\left(S^\perp, S\right)$ and $D \in \mathcal{B}\left(S^\perp\right)$ are arbitrary.

Proof. The argument for the positive operators is straightforward and will be omitted. In each case, both B and D are easily seen to be arbitrary. The fact that the operator in $\mathcal{B}\left(S, S^\perp\right)$ must be zero follows from the fact that K_S is solid and that we could construct a counterexample if the operator were nonzero.

The difficulty for the $\mathbf{Z}$-operators is in showing that $A \in \mathbf{Z}\left(K_S\right)$ necessarily. This however follows from the relationship,

$$(x, s) \in C\left(K_S\right) \iff \left((x,0)^T, (s,t)^T\right) \in C\left(K_S \times \{0\}_{S^\perp}\right) \text{ for all } t \in S^\perp.$$

The result for $\mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)$ then follows from $\mathbf{LL}(H) = \operatorname{linspace}\left(\mathbf{Z}(H)\right)$. ☐

Lemma 5. *If K_V is a cone-space pair and if $S = \operatorname{span}(K)$, then*

$$\beta\left(K_V\right) = \beta\left(K_S\right) + \operatorname{codim}\left(K_V\right) \cdot \dim\left(V\right).$$

Proof. Using Proposition 29, we work with $K_S \times \{0\}_{S^\perp}$ instead of K_V. For $L \in \mathbf{LL}\left(K_S \times \{0\}_{S^\perp}\right)$, we are free to choose A, B, and D in Proposition 30 from their respective spaces having dimensions $\beta\left(K_S\right)$, $\dim\left(S^\perp\right)\dim\left(S\right)$, and $\dim\left(S^\perp\right)^2$. Thus,

$$\beta\left(K_V\right) = \beta\left(K_S\right) + \dim\left(S^\perp\right)\left(\dim\left(S\right) + \dim\left(S^\perp\right)\right).$$ ☐

Using duality we have an immediate consequence.

Lemma 6. *If K_V is a cone-space pair and if $P = \operatorname{span}(K_V^*)$, then*

$$\beta(K_V) = \beta(K_P) + \operatorname{lin}(K_V) \cdot \dim(V).$$

Proof. Apply Lemma 5 to K_V^* to obtain

$$\beta(K_V^*) = \beta((K_V^*)_P) + \operatorname{codim}(K_V^*) \cdot \dim(V).$$

Now $(K_V^*)_P$ is solid and, by Proposition 28, equal to K_P^*. Take its dual and apply Proposition 2 to show that K_P is pointed. Substitute $\beta(K_V^*) = \beta(K_V)$ and $\beta(K_P^*) = \beta(K_P)$ by Proposition 23 to obtain the result. □

The preceding lemmata combine to handle any closed convex cone.

Theorem 9. *If K_V is a cone-space pair with $S \coloneqq \operatorname{span}(K)$ and $P \coloneqq \operatorname{span}(K_S^*)$, then K_{SP} is proper and*

$$\beta(K_V) = \beta(K_{SP}) + \operatorname{lin}(K) \cdot \dim(K) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Proof. Apply Lemma 5 to K_V so that we have

$$\beta(K_V) = \beta(K_S) + \operatorname{codim}(K_V) \cdot \dim(V) \tag{3.1}$$

where K_S is solid. Now apply Lemma 6 to K_S:

$$\beta(K_S) = \beta((K_S)_P) + \operatorname{lin}(K_S) \cdot \dim(S) \tag{3.2}$$

where $(K_S)_P = K_{SP}$ is pointed. The lineality of K_S and dimension of S are the same as those of K itself, so combining (3.1) and (3.2), we have

$$\beta(K_V) = \beta(K_{SP}) + \operatorname{lin}(K) \cdot \dim(K) + \operatorname{codim}(K_V) \cdot \dim(V).$$

Since K_S was solid, the cone-space pair K_{SP} is solid (thus proper) as well. □

The literature states that id_V is Lyapunov-like on any proper cone-space pair K_V, and that therefore $\beta(K_V) \geq 1$. However, the trivial cone in the trivial space is both solid and pointed with Lyapunov rank zero. We caution that the K_{SP} obtained in Theorem 9 can be trivial, as our next example shows.

Example 21. If $K = \mathbb{R}^m$ in $V = \mathbb{R}^n$, then $\operatorname{lin}(K) = \dim(K) = m$, $\operatorname{codim}(K_V) = n - m$, and K_{SP} is trivial. Theorem 9 then gives $\beta(K_V) = n^2 - m(n - m)$.

Example 22. If $K = \operatorname{cone}(\{v\})$ in the n-dimensional space V (cf. Proposition 26), then $\operatorname{lin}(K) = 0$, $\dim(K) = 1$, and $\operatorname{codim}(K_V) = n - 1$. If $S = \operatorname{span}(\{v\})$, then the solid cone-space pair K_S is just $\operatorname{cone}(\{v\})_S$ which is self-dual in S. As a result, $P = \operatorname{span}(K_S^*) = S$ and so $K_{SP} = K_S$. Corollary 7 and Theorem 9 give $\beta(K_V) = n^2 - n + 1$.

Example 23. If K_V is proper, then $S = P = V$, so $K_{SP} = K_V$ and $\operatorname{lin}(K) = \operatorname{codim}(K_V) = 0$. Theorem 9 reduces to $\beta(K_V) = \beta(K_{SP})$.

Theorem 9 simplifies the computation of $\beta(K_V)$ when K_V is improper. Every Lyapunov rank computation reduces to that of a proper cone-space pair K_{SP}.

Theorem 10 (Gowda and Tao)**.** *For every proper polyhedral cone-space pair K_V in $V = \mathbb{R}^n$, we have $1 \leq \beta(K_V) \leq n$ and $\beta(K_V) \neq n - 1$.*

As we noted subsequent to Theorem 9, the trivial cone-space pair is proper and polyhedral, so we must admit $0 \leq \beta(K_V) \leq n$ if we allow n to be zero. Doing so, we extend Theorem 10 to all cone-space pairs.

Theorem 11. *If K_V is a polyhedral cone-space pair, then $\beta(K_V) \neq \dim(V) - 1$.*

Proof. If $\dim(K) = m$, $\dim(V) = n$, and $\operatorname{lin}(K) = \ell$, then from Theorem 9,

$$\beta(K_V) = \beta(K_{SP}) + n^2 + m(\ell - n).$$

Now we set $\beta(K_V) = n - 1$, and rule out all three cases for $\beta(K_{SP})$.

Case 1: $m = n$ **and** $\ell = 0$**.**

This gives $\beta(K_{SP}) = n - 1$ which is impossible by Theorem 10.

Case 2: $m = n$ **and** $\ell > 0$**.**

Since $\ell \in \mathbb{Z}$, we find $\beta(K_{SP}) = n - 1 - n\ell < -1$.

Case 3: $m < n$**.**

We maximize $\beta(K_{SP}) = n - 1 - n^2 + m(n - \ell)$ over ℓ and m by setting $\ell = 0$ and $m = n-1$. Doing so, we reach the impossible conclusion that $\beta(K_{SP}) \leq -1$. $\square$

3.3 A polyhedral Lyapunov rank algorithm

For polyhedral cone-space pairs, some generating set—and therefore the associated discrete complementarity set—is finite. This allows us to compute both $\mathbf{LL}(K_V)$ and $\beta(K_V)$. Our first algorithm computes $\mathbf{LL}(K_V)$ for any polyhedral cone-space pair K_V. It is based on the codimension formula (2.7); recall:

$$L \in \mathbf{LL}(K_V) \iff \langle s \otimes x, L \rangle = 0 \text{ for all } (x, s) \in C(K_V) \cap (G_1 \times G_2). \tag{3.3}$$

Given matrix representations for $s \otimes x$ and L, the trace inner product $\langle s \otimes x, L \rangle$ is equal to $\langle \operatorname{vec}(s \otimes x), \operatorname{vec}(L) \rangle$. We leverage this to compute $\mathbf{LL}(K_V)$: finding all L

satisfying (3.3) becomes the computation of an orthogonal complement.

Algorithm 3 Compute a basis for $\mathbf{LL}(K_V)$

Input: A cone-space pair K_V

Output: A basis for $\mathbf{LL}(K_V)$

function LL(K_V)

$G_1 \leftarrow$ a minimal set of generators for K_V

$G_2 \leftarrow$ a minimal set of generators for K_V^* ▷ obtainable from G_1

$C \leftarrow \{(x, s) \mid x \in G_1, s \in G_2, \langle x, s\rangle = 0\}$ ▷ discrete complementarity set

$W \leftarrow \{\mathrm{vec}(s \otimes x) \mid (x, s) \in C\}$

$B \leftarrow$ a basis for $W^\perp$ ▷ computed via e.g. Gram-Schmidt

return $\{\mathrm{mat}(b) \mid b \in B\}$

end function

If $\dim(V) = n$, then K will be input as a list of generators—essentially elements of $\mathbb{Q}^n$, or n-tuples of rational numbers. The arithmetic in Algorithm 3 should be exact, so in general it is not possible to normalize the generators that arise.

We have a way to compute $\beta(K_V)$: simply call LL(K_V) and count how many elements we get back. In fact this is the best algorithm known for proper cones. But if K_V is not guaranteed to be proper, Theorem 9 provides a more efficient approach. To use Theorem 9, we need to implement the "restrict to subspace" map $K_V \mapsto K_W$. Existing routines assume the dimension of V based on the length n of the input generators, and make no provision for operating in a subspace (reminiscent of why we introduced cone-space pairs). The difficulty is described in greater detail by Orlitzky [27]. We simply present the algorithm, remarking that it is accurate only up to a change of basis. By Proposition 24, the change of basis is irrelevant.

Algorithm 4 Restrict K_V to W (up to a change of basis)

Input: A cone-space pair K_V and a subspace W of dimension m containing K

Output: A new cone-space pair $J_{\mathbb{Q}^m}$ such that $\beta\left(J_{\mathbb{Q}^m}\right) = \beta\left(K_W\right)$

```
function RESTRICT_TO_SPACE(K_V, W)
    B ← a basis for W
    G ← a minimal set of generators for K_V
    J ← ∅
    for x ∈ G do
        w ← the B-coordinates of x          ▷ disregarding coordinates for B^⊥
        J ← J ∪ {w}
    end for
    return cone(J)_{Q^m}
end function
```

We next describe an efficient algorithm for calculating the Lyapunov rank of a cone-space pair K_V. There are three expensive steps in Algorithm 3. The first is the computation of the generators of K_V^*. The standard approach uses the facet normals of K_V, and that problem grows with the number of generators of K_V. The second expensive operation is the combinatoric construction of the discrete complementarity set. Finally, there is the basis computation using a relative of Gram-Schmidt. Since the basis consists of vectorized $n \times n$ matrices, that takes place in $\mathbb{Q}^{n^2}$.

Algorithm 5 is often an improvement over Algorithm 3 because each of those problems is reduced in size. The proper cone-space pair K_{SP} in Theorem 9 will—in general—have fewer generators, fewer facets, fewer complementary pairs, and live in a space of smaller dimension than K_V. It is therefore easier to compute the generators of K_{SP}^* than it is for K_V^*. Moreover, the discrete complementarity set of K_{SP} is constructed from two smaller sets than that of K_V. Finally, if K_{SP} lives in a space of dimension $m < n$, then the basis computation takes place over $\mathbb{Q}^{m^2}$ rather than $\mathbb{Q}^{n^2}$. To be fair, we must now compute $\dim(K)$ and $\operatorname{lin}(K)$, and we potentially call RESTRICT_TO_SPACE() twice. However, those computations are relatively fast. The efficient algorithm is implemented in the Sage Mathematics [39] system.

Algorithm 5 Compute the Lyapunov rank of K_V

Input: A cone-space pair K_V

Output: The Lyapunov rank of K_V

function BETA(K_V)

  $\beta \leftarrow 0$; $n \leftarrow \dim(V)$; $m \leftarrow \dim(K)$; $\ell \leftarrow \operatorname{lin}(K)$

  if $m < n$ **then**

    $K_V \leftarrow$ RESTRICT_TO_SPACE $(K_V, \operatorname{span}(K_V))$

    $\beta \leftarrow \beta + (n - m)\,n$ ▷ Lemma 5

  end if

  if $\ell > 0$ **then**

    $K_V \leftarrow$ RESTRICT_TO_SPACE $(K_V, \operatorname{span}(K_V^*))$

    $\beta \leftarrow \beta + \ell m$ ▷ Lemma 6

  end if

  return $\beta + \operatorname{card}(\operatorname{LL}(K_V))$ ▷ K_V is proper here, so do it the hard way

end function

3.4 The exponential connection

We restate a theorem of Schneider and Vidyasagar [35] in more general terms.

Theorem 12. *If K is a proper cone in a finite-dimensional real Hilbert space V and if $L \in \mathcal{B}(V)$, then $L \in \mathbf{Z}(K)$ if and only if $e^{-tL} \in \pi(K)$ for all $t \geq 0$.*

This theorem has been used effectively. Elsner [10] equates exponentially-positive, resolvent-positive, essentially-positive, quasimonotone, and cross-positive operators. Damm [8] shows that Lyapunov-like operators on $\mathcal{S}^n_+$ and $\mathcal{H}^n_+$ are the familiar Lyapunov transformations from dynamical systems. Gowda and Tao [19] characterize the Lie algebra of the automorphism group of a cone. We prove the general version in two steps.

Lemma 7. *If K_V is a cone-space pair and if $L \in \mathcal{B}(V)$, then $e^{-tL} \in \pi(K_V)$ for all $t \geq 0$ implies that $L \in \mathbf{Z}(K_V)$.*

Proof. Let $e^{-tL} \in \pi(K_V)$ for all $t \geq 0$, and take any $(x, s) \in C(K_V)$. We show that $\langle L(x), s \rangle \leq 0$ and it follows that $L \in \mathbf{Z}(K_V)$. Since $e^{-tL}(x) \in K$,

$$\frac{1}{t}\left\langle \left[e^{-tL} - \mathrm{id}_V\right](x), s\right\rangle = \frac{1}{t}\left\langle e^{-tL}(x), s\right\rangle \geq 0 \text{ for all } t > 0.$$

Take the limit as $t \to 0$ to find $\langle L(x), s \rangle \leq 0$. □

To prove the other inclusion, we will ultimately rely on Theorem 12 for proper cones. To do that, we'll use the proper subcone that appears in Theorem 9. Recall that the cone-space pair $K_S \times \{0\}_{S^\perp}$ is obtained from K_V by isometry. The following propositions relate its positive and $\mathbf{Z}$-operators to those of K_V.

Proposition 31. *If K_V is a cone-space pair and if ψ is an isometry, then we have $\mathbf{Z}\left(\psi\left(K_V\right)\right) = \psi \mathbf{Z}\left(K_V\right)\psi^{-1}$ and $\pi\left(\psi\left(K_V\right)\right) = \psi\pi\left(K_V\right)\psi^{-1}$.*

Proposition 32. *If K_V is a cone-space pair, then $L \in \mathbf{Z}\left(K_V\right)$ if and only if $L^* \in \mathbf{Z}\left(K_V^*\right)$, and $L \in \pi\left(K_V\right)$ if and only if $L^* \in \pi\left(K_V^*\right)$.*

The next few results constitute the converse of Lemma 7. The first demonstrates the converse for the pointed cone-space pair $K_S \times \{0\}_{S^\perp}$. We then do away with the isometry and show that the converse holds for any pointed cone-space pair. Using duality, we show the same for solid cone-space pairs, and the general case follows as a consequence.

Proposition 33. *If K_V is a pointed cone-space pair, then $L \in \mathbf{Z}\left(K_S \times \{0\}_{S^\perp}\right)$ implies that $e^{-tL} \in \pi\left(K_S \times \{0\}_{S^\perp}\right)$ for all $t \geq 0$.*

Proof. Consider Proposition 30 and expand $L \in \mathbf{Z}\left(K_S \times \{0\}_{S^\perp}\right)$,

$$e^{-tL} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^k = \begin{bmatrix} e^{-tA} & \widetilde{B} \\ 0 & e^{-tD} \end{bmatrix}.$$

Note that K_S is pointed and thus proper, so Theorem 12 applied to A and e^{-tA} gives the result by Proposition 30. $\square$

This result extends to any pointed cone-space pair through Proposition 31.

Proposition 34. *If K_V is a pointed cone-space pair, then $L \in \mathbf{Z}\left(K_V\right)$ implies that $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$.*

Proof. Write $\phi\left(K_V\right) = K_S \times \{0\}_{S^\perp}$ where ϕ is an isometry and K_S is proper. Take any $L \in \mathbf{Z}\left(K_V\right)$. Then $\phi L \phi^{-1} \in \mathbf{Z}\left(K_S \times \{0\}_{S^\perp}\right)$ by Proposition 31, and $e^{-t\phi L\phi^{-1}} =$

$\phi e^{-tL}\phi^{-1} \in \pi\left(K_S \times \{0\}_{S^\perp}\right)$ for all $t \geq 0$ by Proposition 33. Use Proposition 31 again to obtain $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$. $\square$

It also extends to solid cone-space pairs through Proposition 32.

Proposition 35. *If K_V is a solid cone-space pair, then $L \in \mathbf{Z}\left(K_V\right)$ implies that $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$.*

Proof. The cone-space pair K_V^* is pointed by Proposition 2. Apply Proposition 34 to K_V^* to obtain $L^* \in \mathbf{Z}\left(K_V^*\right)$ if and only if $e^{-t(L^*)} \in \pi\left(K_V^*\right)$ for all $t \geq 0$. Now apply Proposition 32 to both expressions. $\square$

The result for a solid cone-space pair is all we need for the general case.

Lemma 8. *If K_V is a cone-space pair, then $L \in \mathbf{Z}\left(K_V\right)$ implies that $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$.*

Proof. The proof of Proposition 33 relies on the fact that K_S is proper, or that K_V is pointed. However, using Proposition 35, we can prove Proposition 33 and Proposition 34 without the assumption that K_V and (thus) K_S are pointed. $\square$

Theorem 13. *If K_V is a cone-space pair and if $L \in \mathcal{B}\left(V\right)$, then $L \in \mathbf{Z}\left(K_V\right)$ if and only if $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$.*

Proof. Combine Lemma 7 and Lemma 8. $\square$

A similar result appears in Hilgert, Hofmann, and Lawson [21]. The first two items of their Theorem III.1.9 state that $L \in \mathbf{Z}\left(K_V\right)$ if and only if $e^{-tL} \in \pi\left(K_V\right)$ for all $t \geq 0$. However, the remaining items suggest some hidden assumptions, and its proof relies on another Theorem I.5.27 where the cone is solid. Nevertheless, their Theorem I.5.17 seems to provide the machinery needed to prove the result.

Examples 7, 8, 16, and 17 corroborate Theorem 13 in simple cases. The next provides an application to dynamical systems.

Example 24. The system $x'(t) = -L(x(t))$ has solution $x(t) = e^{-tL}(x(0))$. If $L \in \mathbf{Z}(K_V)$ for some cone-space pair K_V, then Theorem 13 shows that $e^{-tL} \in \pi(K_V)$ for all $t \geq 0$. Therefore, $x(t)$ remains in K for $t > 0$ if $x(0) \in K$.

Recall that $\operatorname{linspace}(\mathbf{Z}(K_V)) = \mathbf{LL}(K_V)$. By applying Theorem 13 to both L and $-L$, we extend the a result of Gowda and Tao [19] to all closed convex cones. The equivalence $L \in \operatorname{Lie}(G) \iff e^{tL} \in G$ for all $t \in \mathbb{R}$ is a known property of matrix groups discussed in Section 7.6 of Baker [3].

Theorem 14. *If K_V is a cone-space pair and if $L \in \mathcal{B}(V)$, then the following are equivalent:*

- *L is Lyapunov-like on K_V.*
- *$e^{tL} \in \operatorname{Aut}(K_V)$ for all $t \in \mathbb{R}$.*
- *$L \in \operatorname{Lie}(\operatorname{Aut}(K_V))$.*

Thus, $\mathbf{LL}(K_V)$ is the Lie algebra of $\operatorname{Aut}(K_V)$. Hilgert, Hofmann, and Lawson derive the same result from their Theorem III.1.9, but the aforementioned caveats still apply. When $K = V = \mathbb{R}^n$, this confirms the fact that the $n \times n$ real matrices are the Lie algebra of the general linear group of degree n over $\mathbb{R}$.

Corollary 8. *If K_V is a cone-space pair, then $\beta(K_V) = \dim(\operatorname{Lie}(\operatorname{Aut}(K_V)))$.*

Chapter 4

Bounding the Lyapunov rank of a proper cone

4.1 An improved upper bound

Since $\beta(K) \coloneqq \dim(\mathbf{LL}(K))$, we have $\beta(K) \geq m$ if we can exhibit m linearly-independent Lyapunov-like operators on K. In determining $\beta(K)$, it is therefore useful to have an upper bound on its values; if one can achieve the upper bound, then the bound is equal to $\beta(K)$. Certainly we have the bound $\beta(K) \leq \dim(\mathcal{B}(V))$ in the ambient space V, but this can be improved when K is proper.

Notice from the codimension formula (2.7) that $\mathbf{LL}(K)^{\perp}$ consists of operators of the form $s \otimes x$ where $(x, s) \in C(K)$. By definition we have $\beta(K) = \dim(\mathcal{B}(V)) - \dim\left(\mathbf{LL}(K)^{\perp}\right)$, so if we can construct m linearly-independent $s \otimes x$ from $C(K)$, then we improve the upper bound to $\beta(K) \leq \dim(\mathcal{B}(V)) - m$. In this manner, Gowda and Tao [19] showed that $\beta(K) \leq \dim(\mathcal{B}(V)) - \dim(V)$.

Theorem 15 (Gowda and Tao)**.** *If K is a proper cone in a real Hilbert space V of dimension $n \geq 2$, then $1 \leq \beta(K) \leq n^2 - n$.*

Proof. We supply the proof of Gowda and Tao because we will use the details later. Since K is proper, Fenchel's Theorem 13 guarantees that K is generated by its boundary [11]. There must exist n linearly-independent vectors on $\operatorname{bdy}(K)$, because otherwise, $\dim(K) < n$. An invertible change-of-basis operator sends those

n vectors to an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ of V, so by Proposition 24, we suppose that $\{x_1, x_2, \ldots, x_n\} \subseteq \operatorname{bdy}(K)$.

Proposition 5 implies that for each $x_i \in \operatorname{bdy}(K)$, there exists an associated nonzero $s_i \in \operatorname{bdy}(K^*)$ such that $(x_i, s_i) \in C(K)$. Define A_i to be $s_i \otimes x_i$. Then $\{A_i\}_{i=1}^n$ is a pairwise-orthogonal n-element subset of $\mathbf{LL}(K)^\perp$. $\square$

If $\dim(V) = n$, we are able to improve this upper bound to $\beta(K) \leq (n-1)^2$. The construction of the first n elements of $\mathbf{LL}(K)^\perp$ follows Gowda and Tao; we then proceed to construct an additional $n-1$ operators for a total of $2n-1$.

When K is polyhedral, we have $\beta(K) \leq n$ and that bound is tight [19]. We can therefore assume that K is non-polyhedral. Since all closed convex cones are polyhedral when $\dim(V) = n \leq 2$, we will assume that $n \geq 3$.

Lemma 9. *If K is a proper cone in a real Hilbert space V of dimension $n \geq 2$ and if $\operatorname{bdy}(K)$ is contained in a finite union of hyperplanes $\bigcup_{i=1}^m H_i$, then K is polyhedral.*

Proof. A hyperplane H is a supporting hyperplane to K if $H \cap K \subseteq \operatorname{bdy}(K)$ and $H \cap K \neq \emptyset$. Our proof uses induction on the number of hyperplanes H_i which *do not* support K. No generality is lost by ignoring the H_i such that $H_i \cap K = \emptyset$.

Case 1: base case**.**

In this case, each H_i supports K and defines a half-space G_i containing K. The cone $C \coloneqq \bigcap_{i=1}^m G_i$ is polyhedral [31]. We claim that $K = C$.

Assume on the contrary that $K \neq C$. Since K is proper and $n \geq 2$, every point of $\operatorname{int}(K)$ is a convex combination of two points in $\operatorname{bdy}(K)$. We therefore assume that there is an $x \in \operatorname{bdy}(C)$ such that $x \notin K$. Suppose $y \in \operatorname{int}(K) \subseteq \operatorname{int}(C)$, and define $S := \{(1-\alpha)\, y + \alpha x \mid 0 \le \alpha < 1\}$. Since C and K are closed and convex [31], we have $S \subseteq \operatorname{int}(C)$, and there exists an $r \in S$ such that $r \in \operatorname{bdy}(K)$.

Now $\operatorname{bdy}(K) \subseteq \bigcup_{i=1}^{m} H_i$ by assumption, and $r \in \operatorname{bdy}(K)$ implies that $r \in H_i$ for some i. We should have $r \in S \subseteq \operatorname{int}(C)$, yet $r \notin \operatorname{int}(C)$ because H_i is a *supporting* hyperplane to C. This contradiction shows that $K = C$ is polyhedral.

Case 2: inductive case**.**

Some H_i, which we call $\widetilde{H}$, is not a supporting hyperplane to K and must therefore intersect the interior of K. This $\widetilde{H}$ splits K into two proper cones K_1 and K_2, each contained in one of the two half-spaces defined by $\widetilde{H}$. Both K_1 and K_2 are proper, since K was. But now $\widetilde{H}$ is a supporting hyperplane to both K_1 and K_2, so they both have fewer non-supporting hyperplanes than K itself. Apply the induction hypothesis to conclude that K_1 and K_2 are polyhedral. As such, K_1 and K_2 are finitely-generated. Let,

$$K_1 := \operatorname{cone}(\{x_1, x_2, \ldots, x_\ell\})\,; \quad K_2 := \operatorname{cone}(\{y_1, y_2, \ldots, y_k\})\,.$$

By appealing to the convexity of K, it is easy to see that

$$K = K_1 \cup K_2 = \operatorname{cone}(\{x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_k\})\,.$$

Hence, K is polyhedral. $\square$

We will use this lemma to ensure the existence of linearly-independent vectors on the boundary of K. Those vectors will in turn be used to find pairs $(x, s) \in C(K)$ that form linearly-independent elements $s \otimes x \in \mathbf{LL}(K)^{\perp}$.

Theorem 16. *If K is a proper cone in a real Hilbert space V of dimension $n \geq 3$, then $1 \leq \beta(K) \leq (n-1)^2$.*

Proof. Let x_i, s_i, and A_i for $i \in \{1, 2, \ldots, n\}$ be as in the proof of Theorem 15. We assume that $\{s_i\}_{i=1}^n$ is linearly-independent, because that will turn out to be the worst case. As a result, the s_i give us n distinct points on the boundary of K^*.

Now, there exists a nonzero $b_1 \in \operatorname{bdy}(K^*)$ such that $b_1 \notin \bigcup_{i=1}^n \operatorname{span}(\{s_i\})$, because $\operatorname{bdy}(K^*)$ does not consist of only n rays. By Proposition 5, we can find a nonzero $a_1 \in \operatorname{bdy}(K)$ with $\langle a_1, b_1 \rangle = 0$; that is, with $(a_1, b_1) \in C(K)$. We define a new operator $B_1 \coloneqq b_1 \otimes a_1$, and claim that the set $\{B_1\} \cup \{A_i\}_{i=1}^n$ is linearly-independent in $\mathcal{B}(V)$. To see this, suppose that $B_1 = \sum_{i=1}^n \alpha_i A_i$. Note that

$$A_i(x_j) = \langle x_i, x_j \rangle s_i = \begin{cases} s_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha_j \neq 0$ for some index j, then we apply B_1 to x_j:

$$B_1(x_j) = \langle a_1, x_j \rangle b_1 = \left[\sum_{i=1}^n \alpha_i A_i \right](x_j) = \alpha_j s_j.$$

This contradicts either our choice of b_1, or that $s_j \neq 0$.

This procedure can be repeated to produce $b_2, \ldots, b_{n-1}$, but it becomes more difficult with the addition of each successive b_i. We therefore assume that we have $(n-2)$ such vectors $b_1, b_2, \ldots, b_{n-2}$ with $\{b_1, b_2, \ldots, b_{n-2}, s_j\}$ linearly-independent for all j, and proceed to find b_{n-1}. Define the sets,

$$F_j \coloneqq \operatorname{span}(\{b_1, b_2, \ldots, b_{n-2}, s_j\}).$$

Each F_j defines an $(n-1)$-dimensional space, a hyperplane. By Lemma 9, we can find a $b_{n-1} \in \operatorname{bdy}(K^*)$ not contained in any of the F_j. Define $B_k \coloneqq b_k \otimes a_k$ like we defined B_1 previously. We will show that the set

$$\{B_1, B_2, \ldots, B_{n-1}, A_1, \ldots, A_n\} \tag{4.1}$$

is linearly-independent. Suppose that

$$\mu_{n-1} B_{n-1} + \sum_{k=1}^{n-2} \mu_k B_k + \sum_{i=1}^{n} \lambda_i A_i = 0$$

and apply both sides to x_j,

$$\mu_{n-1} \langle a_{n-1}, x_j \rangle \, b_{n-1} + \lambda_j s_j + \sum_{k=1}^{n-2} \mu_k \langle a_k, x_j \rangle \, b_k = 0. \tag{4.2}$$

If $\mu_{n-1} \neq 0$, then this can be rewritten

$$\langle a_{n-1}, x_j \rangle \, b_{n-1} = -\frac{\lambda_j}{\mu_{n-1}} s_j - \sum_{k=1}^{n-2} \frac{\mu_k}{\mu_{n-1}} \langle a_k, x_j \rangle \, b_k \in F_j. \tag{4.3}$$

We chose $b_{n-1} \notin F_j$, so the only solution to (4.3) is when $\langle a_{n-1}, x_j \rangle = 0$. Since $j \in \{1, 2, \ldots, n\}$ was arbitrary, we must have $\langle a_{n-1}, x_j \rangle = 0$ for all j. But $\{x_j\}_{j=1}^{n}$ is a basis for V and $a_{n-1} \neq 0$, so this is a contradiction: we cannot have $\mu_{n-1} \neq 0$. But if $\mu_{n-1} = 0$, then equation (4.2) reduces to

$$\lambda_j s_j + \sum_{k=1}^{n-2} \mu_k \langle a_k, x_j \rangle \, b_k = 0.$$

We have assumed that $\{b_1, b_2, \ldots, b_{n-2}, s_j\}$ is linearly-independent for all j; thus the remaining coefficients are λ_j and μ_k are all zero as well. As a result, the set (4.1) is linearly-independent. There are $2n-1$ elements of $\mathbf{LL}(K)^{\perp}$ in (4.1), so we must have $\beta(K) \leq n^2 - (2n-1) = (n-1)^2$.

The argument fails for b_n. If $\{b_1, b_2, \ldots, b_{n-1}, s_j\}$ is linearly-independent in V, then $\{b_1, b_2, \ldots, b_n, s_j\}$, which contains $n+1$ elements, cannot be. $\square$

4.2 Applications of the bound

We now give two examples where Theorem 16 can be used to deduce the Lyapunov rank of a cone. The first is adapted from Theorem 26 of Rudolf et al. [33].

Example 25. The cone of positive polynomials in $\mathbb{R}^3$ is

$$\mathcal{P}_+^3 := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \;\middle|\; \forall t \in \mathbb{R},\ p(t) = x_1 + x_2 t + x_3 t^2 \geq 0 \right\}.$$

It consists of coefficient vectors of all nonnegative polynomials $p(t)$ with $\deg(p) \leq 2$. Its dual can be expressed [4, 24, 26] in terms of the *moment cone* in $\mathbb{R}^3$,

$$\mathcal{M}^3 := \operatorname{cone}\left(\left\{ \left(1, t, t^2\right)^T \;\middle|\; t \in \mathbb{R} \right\} \right); \quad \left(\mathcal{P}_+^3\right)^* = \operatorname{cl}\left(\mathcal{M}^3\right).$$

Note immediately that any $x = (x_1, x_2, x_3)^T \in \mathcal{P}_+^3$ has $x_3 \geq 0$ since $p(t) \geq 0$ for large t. And any root of $p(t)$ must be a double root: consider its graph, which lies on or above the x-axis. If t_0 is a (double) root of $p(t)$, then $p'(t_0) = 0$.

The extreme directions of $\left(\mathcal{P}_+^3\right)^*$ are known. Rudolf et al. [33] cite sections 2.2 and 6.6 of Karlin and Studden [24] for the following:

$$\operatorname{Ext}\left(\operatorname{cl}\left(\mathcal{M}^3\right)\right) = \left\{ \alpha \left(1, t, t^2\right)^T \;\middle|\; \alpha > 0, t \in \mathbb{R} \right\} \cup \left\{ (0, 0, \alpha)^T \;\middle|\; \alpha \geq 0 \right\}.$$

We show that the following four operators are Lyapunov-like on $\mathcal{P}_+^3$:

$$L_1 := I,\ L_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ L_3 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \text{ and } L_4 := \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

To do that, it suffices by Proposition 22 to check the discrete complementarity set $C\left(\mathcal{P}_+^3\right) \cap \left(\operatorname{Ext}\left(\mathcal{P}_+^3\right) \times \operatorname{Ext}\left(\operatorname{cl}\left(\mathcal{M}^3\right)\right)\right)$. The identity is trivially Lyapunov-like. Suppose that $x = (x_1, x_2, x_3)^T \in \mathcal{P}_+^3$ and $s \in \operatorname{Ext}\left(\operatorname{cl}\left(\mathcal{M}^3\right)\right)$ with $\langle x, s \rangle = 0$.

Case 1: $s = (0, 0, \alpha)^T$.

The equation $\langle x, s\rangle = 0$ implies that $x_3 = 0$. If $x_2 \neq 0$, then $p(t)$ has odd degree and cannot be nonnegative for all $t \in \mathbb{R}$. Thus $x = (x_1, 0, 0)^T$. Compute each inner product and verify that $\langle L_i(x), s\rangle = 0$.

Case 2: $s = \alpha (1, t_0, t_0^2)^T$.

Here $\langle x, s\rangle = 0$ implies that t_0 is a double root of $p(t) = x_1 + x_2 t + x_3 t^2$. Thus,

$$(1/\alpha) \langle L_2(x), s\rangle = x_2 + 2x_3 t_0 = p'(t_0) = 0,$$

$$(1/\alpha) \langle L_3(x), s\rangle = 2x_1 + x_2 t_0 = 2p(t_0) - t_0 p'(t_0) = 0 - 0 = 0, \text{ and}$$

$$(1/\alpha) \langle L_4(x), s\rangle = 2x_1 t_0 + x_2 t_0^2 = 2t_0 p(t_0) - t_0^2 p'(t_0) = 0 - 0 = 0.$$

In both cases, we have exhibited four linearly-independent elements of $\mathbf{LL}\left(\mathcal{P}_+^3\right)$, so we must have $\beta\left(\mathcal{P}_+^3\right) = \dim\left(\mathbf{LL}\left(\mathcal{P}_+^3\right)\right) \geq 4$. But $\beta\left(\mathcal{P}_+^3\right) \leq (n-1)^2 = 4$ by Theorem 16, implying equality.

This example inspires a few corollaries.

Corollary 9. *The upper bound $\beta(K) \leq (n-1)^2$ is tight when $n = \dim(V) = 3$, and each possible value $\beta(K) \in \{1, 2, 3, 4\}$ is achieved for some cone K in V.*

Proof. The upper bound is tight for $n = 3$ since it is achieved by $\mathcal{P}_+^3$. Other cones in $\mathbb{R}^3$ with Lyapunov ranks 1, 2, 3, and 4 are known [19]. □

Every symmetric cone is perfect [19], and in nontrivial spaces, there always exist symmetric cones whose Lyapunov rank exceeds the dimension of the ambient space. We show that asymmetric cones with the same property exist.

Corollary 10. *For every integer $n \geq 3$, there exists an asymmetric cone K in an n-dimensional Hilbert space having $\beta(K) > n$.*

Proof. Define the cone $K \coloneqq \mathcal{P}_{+}^{3} \times \mathbb{R}_{+}^{n-3}$ in the space $\mathbb{R}^{3} \times \mathbb{R}^{n-3}$. K is asymmetric since $K^{*} = \mathcal{M}^{3} \times \mathbb{R}_{+}^{n-3} \neq K$. Lyapunov rank is additive on proper cones by Proposition 27, so $\beta(K) = n + 1$ and $\dim(\mathbb{R}^{3} \times \mathbb{R}^{n-3}) = n$. □

Chapter 5

Linear games on proper cones

In this chapter, we present a generalization of a two-person zero-sum matrix game [23, 29, 40]. Classically, such a game involves a matrix $A \in \mathbb{R}^{n \times n}$ and the set of "strategies" $\Delta \coloneqq \operatorname{conv}(\mathbf{e})$ from which two players are free to choose. If the players choose $x, y \in \Delta$, respectively, then the game is played by evaluating $y^T A x$ as the "payoff" to the first player. The payoff to the second player is $-y^T A x$.

Each player will try to maximize his payoff in this scenario, or—what is equivalent—try to minimize the payoff of his opponent. In fact, the existence of optimal strategies is guaranteed for both players [23]. The *value* of the matrix game A is the payoff resulting from optimal play,

$$v(A) \coloneqq \max_{x \in \Delta} \ \min_{y \in \Delta} \left(y^T A x\right) = \min_{y \in \Delta} \ \max_{x \in \Delta} \left(y^T A x\right).$$

The payoff to the first player in this case is $v(A)$. Corresponding to $v(A)$ is an optimal strategy pair $(\bar{x}, \bar{y}) \in \Delta \times \Delta$ such that

$$\bar{y}^T A x \leq v(A) = \bar{y}^T A \bar{x} \leq y^T A \bar{x} \text{ for all } (x, y) \in \Delta \times \Delta.$$

The relationship between A, $\bar{x}$, $\bar{y}$, and $v(A)$ has been studied extensively [22, 30]. Gowda and Ravindran [15] were motivated by these results to ask if the matrix A can be replaced by a linear transformation L, and whether or not the unit simplex Δ can be replaced by a more general set—a base of a self-dual cone. We extend their generalization to non-self-dual (but still proper) cones.

5.1 Definition of a linear game

In the classical setting, the interpretation of the strategies as probabilities results in a strategy set $\Delta \subseteq \mathbb{R}^n_+$ that is compact, convex, and does not contain the origin. Moreover, any nonzero $x \in \mathbb{R}^n_+$ is a unique positive multiple $x = \lambda b$ of some $b \in \Delta$. Several existence and uniqueness results are predicated on those properties.

Definition 26. Suppose that K is a cone and $B \subseteq K$ does not contain the origin. If any nonzero $x \in K$ can be uniquely represented $x = \lambda b$ where $\lambda > 0$ and $b \in B$, then B is a *base* of K.

The set Δ is a compact convex base for the proper cone $\mathbb{R}^n_+$. Every $x \in \Delta$ also has entries that sum to unity, which can be abbreviated by the condition $\langle x, \mathbf{1} \rangle = 1$ where $\mathbf{1} \coloneqq (1, 1, \ldots, 1)^T$ happens to lie in the interior of $\mathbb{R}^n_+$. This motivates a generalization where $\mathbb{R}^n_+$ is replaced by a proper cone.

Definition 27. Let V be a finite-dimensional real Hilbert space. A *linear game* in V is a tuple (L, K, e, e^*) where $L \in \mathcal{B}(V)$, the set K is a proper cone in V, and the points e and e^* belong to $\operatorname{int}(K)$ and $\operatorname{int}(K^*)$ respectively.

Definition 28. We define the game $\mathcal{G}$ and its *dual game* $\mathcal{G}^*$ by

$$\mathcal{G} \coloneqq (L, K, e, e^*), \qquad \mathcal{G}^* \coloneqq (L^*, K^*, e^*, e).$$

The conditions on V, L, K, e, and e^* are all implicit when we refer to a linear game (L, K, e, e^*). The game $\mathcal{G}$ is intended to introduce the names L, K, e, and e^* under those conditions in the statements of our theorems.

Definition 29. The strategy sets for the game $\mathcal{G}$ are,

$$\Delta_1(\mathcal{G}) := \{x \in K \mid \langle x, e^* \rangle = 1\}; \quad \Delta_2(\mathcal{G}) := \{y \in K^* \mid \langle y, e \rangle = 1\}.$$

Since $e \in \operatorname{int}(K)$ and $e^* \in \operatorname{int}(K^*)$, these are bases for their respective cones. In the context of the game $\mathcal{G}$ we will usually omit the argument $\mathcal{G}$ and write Δ_i to mean $\Delta_i(\mathcal{G})$. However, in a few cases where multiple games are in scope, the dependence of Δ_1 and Δ_2 on the game data is unavoidable.

Definition 30. The *payoff operator* for the game $\mathcal{G}$ is $(x, y) \mapsto \langle L(x), y \rangle$.

To play the game $\mathcal{G}$, the first player chooses an $x \in \Delta_1$, and the second player independently chooses a $y \in \Delta_2$. This completes the turn, and the payoffs are determined by applying the payoff operator to (x, y). The payoff to the first player is $\langle L(x), y \rangle$, and the payoff to the second player is $-\langle L(x), y \rangle$.

The payoff operator is continuous in both arguments because it is bilinear and the ambient space is finite-dimensional. We constructed the strategy sets Δ_1 and Δ_2 to be compact and convex; as a result, a general min-max principle (originally due to von Neumann and appearing as Theorem 1.5.1 in Karlin [23]) guarantees the existence of optimal strategies for both players.

Definition 31. A pair $(\bar{x}, \bar{y}) \in \Delta_1 \times \Delta_2$ is an *optimal pair* for the game $\mathcal{G}$ if it satisfies the *saddle-point inequality*,

$$\langle L(x), \bar{y} \rangle \leq \langle L(\bar{x}), \bar{y} \rangle \leq \langle L(\bar{x}), y \rangle \text{ for all } (x, y) \in \Delta_1 \times \Delta_2. \tag{5.1}$$

At an optimal pair, neither player can unilaterally increase his payoff by changing his strategy. The value $\langle L(\bar{x}), \bar{y} \rangle$ is unique by the same min-max theorem.

Definition 32. The *value* of the game $\mathcal{G}$ is $v(\mathcal{G}) := \langle L(\bar{x}), \bar{y} \rangle$, where $(\bar{x}, \bar{y})$ is any optimal pair in the sense of Definition 31.

We thus have an equivalent characterization of a game's value that does not require us to have a particular optimal pair in mind,

$$v(\mathcal{G}) = \max_{x \in \Delta_1} \min_{y \in \Delta_2} \langle L(x), y \rangle = \min_{y \in \Delta_2} \max_{x \in \Delta_1} \langle L(x), y \rangle. \tag{5.2}$$

Example 26. If $K = \mathbb{R}^n_+$ in $V = \mathbb{R}^n$ and $e = e^* = (1, 1, \ldots, 1)^T \in \operatorname{int}(K)$, then $K^* = K$, and $\Delta_1 = \Delta_2 = \Delta$. For any $L \in \mathbb{R}^{n \times n}$, the linear game $\mathcal{G}$ is a two-person zero-sum matrix game. Its payoff is $(x, y) \mapsto y^T L x$, and its value is

$$v(\mathcal{G}) = v(L) = \max_{x \in \Delta} \min_{y \in \Delta} \left(y^T L x \right) = \min_{y \in \Delta} \max_{x \in \Delta} \left(y^T L x \right).$$

5.2 The value of a linear game

Theorem 17. *Let $(\bar{x}, \bar{y}) \in \Delta_1 \times \Delta_2$ for the game $\mathcal{G}$. Then $L^*(\bar{y}) \overset{*}{\preccurlyeq} \nu e^*$ and $L(\bar{x}) \succcurlyeq \nu e$ for some $\nu \in \mathbb{R}$ if and only if $\nu = v(\mathcal{G})$ and $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$.*

Proof. Suppose that $\nu = v(\mathcal{G})$ and $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$. The inequality (5.1) shows that for all $y \in \Delta_2$, we have $\langle L(\bar{x}), y \rangle \geq v(\mathcal{G}) = v(\mathcal{G}) \langle e, y \rangle$. But Δ_2 is a base for K^*, so the inequality holds for every $y \in K^*$. Rearrange it to obtain $\langle L(\bar{x}) - v(\mathcal{G}) e, y \rangle \geq 0$ for all $y \in K^*$. By definition, $L(\bar{x}) - v(\mathcal{G}) e \in (K^*)^* = K$, and we write $L(\bar{x}) \succcurlyeq \nu e$. The proof of the other inequality is similar.

Conversely, suppose that ν, $\bar{x}$, and $\bar{y}$ satisfy $L^*(\bar{y}) \overset{*}{\preccurlyeq} \nu e^*$ and $L(\bar{x}) \succcurlyeq \nu e$. Work backwards to find that $\langle L(x), \bar{y} \rangle \leq \nu$ for all $x \in \Delta_1$ and $\langle L(\bar{x}), y \rangle \geq \nu$ for all $y \in \Delta_2$. Substitute $x = \bar{x}$ and $y = \bar{y}$ to obtain $\nu = \langle L(\bar{x}), \bar{y} \rangle = v(\mathcal{G})$. $\square$

A number of classical results in game theory are based on an analogue of Theorem 17. One states that if you multiply the entries of a payoff matrix by a nonnegative value, then the payoffs are scaled but the optimal strategies remain optimal. Another says that adding a constant to each entry of a payoff matrix increases the value of the corresponding game by that constant.

Corollary 11. *For all $\lambda \geq 0$, the value of the game $(\lambda L, K, e, e^*)$ is $\lambda v(\mathcal{G})$.*

Proof. Let $(\bar{x}, \bar{y})$ be optimal for $\mathcal{G}$ with value $\nu = v(\mathcal{G})$ so that the cone inequalities in Theorem 17 hold. Multiply both sides of each inequality by $\lambda \geq 0$; the inequalities remain true with λL in place of L and $\lambda \nu$ in place of ν. By the converse of the same theorem, $\lambda \nu = \lambda v(\mathcal{G})$ is the value of the game $(\lambda L, K, e, e^*)$. $\square$

Corollary 12. *If $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$ and if $\lambda \in \mathbb{R}$, then $(\bar{x}, \bar{y})$ is optimal for the game $\mathcal{H} \coloneqq (L + \lambda(e \otimes e^*), K, e, e^*)$ and $v(\mathcal{H}) = v(\mathcal{G}) + \lambda$.*

Proof. Note that $\langle L(\bar{x}) + \lambda [e \otimes e^*](\bar{x}), \bar{y} \rangle = v(\mathcal{G}) + \lambda \langle e, \bar{y} \rangle = v(\mathcal{G}) + \lambda$, and let $\nu \coloneqq v(\mathcal{G}) + \lambda$. Now $\bar{x}$, $\bar{y}$, and ν satisfy the conditions of Theorem 17 for the game $\mathcal{H}$, and therefore $(\bar{x}, \bar{y})$ is optimal for $\mathcal{H}$ with $v(\mathcal{H}) = \nu = v(\mathcal{G}) + \lambda$. $\square$

Corollary 13. *If $(\bar{x}, \bar{y})$ is optimal for the game $\mathcal{G}$, then $(\bar{y}, \bar{x})$ is optimal for the game $(-L^*, K^*, e^*, e)$ with value $-v(\mathcal{G})$.*

Proof. Suppose $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$, and negate the cone inequalities of Theorem 17 so that they reverse. If $M = -L^*$, $C = K^*$, and $\nu = -v(\mathcal{G})$, then the game $\mathcal{H} \coloneqq (M, C, e^*, e)$ has $M(\bar{y}) \succcurlyeq_C \nu e^*$ and $M^*(\bar{x}) \preccurlyeq_{C^*} \nu e$. Now $\bar{y} \in \Delta_1(\mathcal{H})$ and

$\bar{x} \in \Delta_2(\mathcal{H})$, so by Theorem 17 again, the pair $(\bar{y}, \bar{x})$ is optimal for $\mathcal{H}$ and its value is $v(\mathcal{H}) = \nu = -v(\mathcal{G})$. □

Theorem 18. *If $(\bar{x}, \bar{y})$ is an optimal pair for the game $\mathcal{G}$, then*

$$L(\bar{x}) - v(\mathcal{G})\, e \succcurlyeq 0 \overset{*}{\preccurlyeq} v(\mathcal{G})\, e^* - L^*(\bar{y})$$

and

$$\langle \bar{y}, L(\bar{x}) - v(\mathcal{G})\, e \rangle = 0 = \langle v(\mathcal{G})\, e^* - L^*(\bar{y}), \bar{x} \rangle .$$

Moreover, $L(\bar{x}) = v(\mathcal{G})\, e$ if $\bar{y} \overset{*}{\succ} 0$ and $L^*(\bar{y}) = v(\mathcal{G})\, e^*$ if $\bar{x} \succ 0$.

Proof. The nonnegativity and orthogonality relations follow from Theorem 17.

Suppose $\bar{y} \overset{*}{\succ} 0$. Then $\langle x, \bar{y} \rangle > 0$ for all nonzero $x \succcurlyeq 0$. In particular, $\langle L(\bar{x}) - v(\mathcal{G})\, e, \bar{y} \rangle > 0$, since $L(\bar{x}) - v(\mathcal{G})\, e \succcurlyeq 0$ from Theorem 17. But we also have $\langle L(\bar{x}) - v(\mathcal{G})\, e, \bar{y} \rangle = 0$, so these statements contradict unless $L(\bar{x}) - v(\mathcal{G})\, e = 0$, or $L(\bar{x}) = v(\mathcal{G})\, e$. The proof of the other implication is similar. □

Proposition 36. *If $L \in \pi(K)$ for the game $\mathcal{G}$, then $v(\mathcal{G}) \geq 0$.*

Proof. If $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$, then $L(\bar{x}) \succcurlyeq 0$, and $v(\mathcal{G}) = \langle L(\bar{x}), \bar{y} \rangle \geq 0$. □

Proposition 37. *$L \in \mathbf{S}(K)$ in the game $\mathcal{G}$ if and only if $v(\mathcal{G}) > 0$.*

Proof. Suppose $L \in \mathbf{S}(K)$ with $L(d) \succ 0$ for $d \succ 0$. From Theorem 17, we have $v(\mathcal{G})\, e^* \overset{*}{\succcurlyeq} L^*(\bar{y})$ implying that

$$\langle v(\mathcal{G})\, e^*, d \rangle \geq \langle L^*(\bar{y}), d \rangle \iff v(\mathcal{G}) \geq \frac{\langle \bar{y}, L(d) \rangle}{\langle e^*, d \rangle} > 0.$$

If instead we assume that $v(\mathcal{G}) > 0$, then Theorem 17 gives $L(\bar{x}) \succcurlyeq v(\mathcal{G})\, e \succ 0$ since both $v(\mathcal{G}) > 0$ and $e \succ 0$. So $\bar{x}$ is a point such that $L(\bar{x}) \succ 0$. However, $\bar{x}$

may lie on the boundary of K. In that case, using the continuity of L, we can find a nearby point $d \succ 0$ such that $L(d) \succ 0$. Therefore $L \in \mathbf{S}(K)$. $\square$

Proposition 38. *$-L^* \in \mathbf{S}(K^*)$ in the game $\mathcal{G}$ if and only if $v(\mathcal{G}) < 0$.*

Proof. Apply Corollary 13 to Proposition 37. $\square$

Proposition 39. *The function $v(\mathcal{G})$ is continuous in L.*

Proof. Use (5.2) to define $h(L) = v(\mathcal{G})$ by

$$h(L) \coloneqq \max_{x \in \Delta_1} \min_{y \in \Delta_2} \langle L(x), y \rangle = \min_{y \in \Delta_2} \max_{x \in \Delta_1} \langle L(x), y \rangle .$$

By the Cauchy-Schwarz inequality, we have $|\langle M(x), y \rangle| \leq \delta(M)$ for all $x \in \Delta_1$ and $y \in \Delta_2$ where

$$\delta(M) \coloneqq \|M\| \max_{x \in \Delta_1, y \in \Delta_2} \|x\| \, \|y\| .$$

From this, deduce two inequalities,

$$h(L+M) = \min_{y \in \Delta_2} \left[\max_{x \in \Delta_1} \left(\langle L(x), y \rangle + \langle M(x), y \rangle \right) \right] \leq h(L) + \delta(M)$$

and

$$h(L+M) = \max_{x \in \Delta_1} \left[\min_{y \in \Delta_2} \left(\langle L(x), y \rangle + \langle M(x), y \rangle \right) \right] \geq h(L) - \delta(M) .$$

It follows that $|h(L+M) - h(L)| \leq \delta(M)$ which goes to zero with $\|M\|$. $\square$

Proposition 40. *Suppose L is invertible and $L^{-1} \in \pi(K)$. Then,*

$$v(\mathcal{G}) = \frac{1}{\langle L^{-1}(e), e^* \rangle} = \frac{1}{\left\langle e, (L^*)^{-1}(e^*) \right\rangle} = v(\mathcal{G}^*)$$

and an optimal pair $(\bar{x}, \bar{y})$ for $\mathcal{G}$ is $\bar{x} = v(\mathcal{G}) L^{-1}(e)$ and $\bar{y} = v(\mathcal{G}) (L^*)^{-1}(e^*)$.

Proof. Since $L^{-1} \in \pi(K)$, we also have $(L^*)^{-1} \in \pi(K^*)$. Define,

$$u := L^{-1}(e), \qquad \bar{x} := u / \langle u, e^* \rangle$$

$$w := (L^*)^{-1}(e^*), \quad \bar{y} := w / \langle w, e \rangle .$$

Since $u \succcurlyeq 0$ and $e^* \overset{*}{\succ} 0$, we have $\langle u, e^* \rangle > 0$ implying that $\bar{x} \succcurlyeq 0$. Likewise, $\bar{y} \overset{*}{\succcurlyeq} 0$. Furthermore, $\langle \bar{x}, e^* \rangle = 1 = \langle \bar{y}, e \rangle$ by construction, so $\bar{x} \in \Delta_1$ and $\bar{y} \in \Delta_2$. If we define $\nu := 1 / \langle u, e^* \rangle = 1 / \langle w, e \rangle$, then $\bar{x}$, $\bar{y}$, and ν satisfy Theorem 17. $\square$

5.3 Completely-mixed games

A completely-mixed matrix game [22] is one in which no pure strategy is chosen with probability zero in any optimal strategy. Thus any optimal strategies $(\bar{x}, \bar{y})$ for the players lie in $\operatorname{int}\left(\mathbb{R}^n_+\right)$. This idea was generalized by Gowda and Ravindran [15] to self-dual cones, and the extension to non-self-dual cones is straightforward.

Definition 33. A game $\mathcal{G}$ is completely-mixed if every optimal pair $(\bar{x}, \bar{y})$ for $\mathcal{G}$ has $\bar{x} \succ 0$ and $\bar{y} \overset{*}{\succ} 0$.

The following lemmata show that two particular line segments through the interior of K^* will intersect its boundary at (at least) two points.

Lemma 10. *If $0 \overset{*}{\prec} \bar{y} \in \Delta_2$ and $\bar{y} \neq u \in V$ with $\langle u, e \rangle = 1$, then there exist $t > 0$ and $s < 0$ in $\mathbb{R}$ such that $(1+t)\bar{y} - tu \in \operatorname{bdy}(K^*)$ and $(1+s)\bar{y} - su \in \operatorname{bdy}(K^*)$.*

Proof for $t > 0$. If $\bar{y} \overset{*}{\succcurlyeq} u$, then $\langle \bar{y} - u, e \rangle = 0$ implies that $\bar{y} = u$ (since $e \succ 0$) contrary to our assumption. So $\bar{y} \overset{*}{\not\succcurlyeq} u$. The ray $\bar{y} + t(\bar{y} - u) = (1+t)\bar{y} - tu$ passes from the interior of K^* (for small t) to its exterior (for large t); at some point the ray intersects $\operatorname{bdy}(K^*)$. The case where $s < 0$ is similar. $\square$

Lemma 11. *If* $0 \overset{*}{\prec} \bar{y} \in \Delta_2$ *and* $0 \neq u \in V$ *with* $\langle u, e \rangle = 0$, *then there exist* $t > 0$ *and* $s < 0$ *in* $\mathbb{R}$ *such that* $\bar{y} - tu \in \operatorname{bdy}(K^*)$ *and* $\bar{y} - su \in \operatorname{bdy}(K^*)$.

Proof for $t > 0$. Since Δ_2 is bounded, $\bar{y} - tu \notin \Delta_2$ for large t. But $\langle \bar{y} - tu, e \rangle = 1$ for all t, so eventually, $\bar{y} \overset{*}{\nsucceq} tu$. The ray $\bar{y} - tu$ passes from the interior of K^* (for small t) to its exterior (for large t); at some point the ray intersects $\operatorname{bdy}(K^*)$. The case where $s < 0$ is similar. $\square$

The next theorem generalizes a result of Gowda and Ravindran [15] which itself generalizes an earlier result of Kaplansky [22]. Note that if $\bar{y} \overset{*}{\succ} 0$ for an optimal $(\bar{x}, \bar{y})$, then $L(\bar{x}) = 0$ from Theorem 18. And since $\langle \bar{x}, e^* \rangle = 1$ requires $\bar{x} \neq 0$, neither $\ker(L)$ nor $\ker(L^*)$ are trivial in that case.

Theorem 19. *If* $v(\mathcal{G}) = 0$ *and* $\bar{y} \overset{*}{\succ} 0$ *for every optimal pair* $(\bar{x}, \bar{y})$, *then*

(i) $\ker(L^*) \cap e^{\perp} = \{0\}$*;*

(ii) $\dim(\ker(L^*)) = \dim(\ker(L)) = 1$*;*

(iii) $L(\bar{x}) = 0$ *and* $L^*(\bar{y}) = 0$ *for any optimal pair* $(\bar{x}, \bar{y})$*;*

(iv) the optimal pair $(\bar{x}, \bar{y})$ *is unique and* $\bar{x} \succ 0$.

Proof of (i). Suppose $u \in \ker(L^*) \cap e^{\perp}$ so that $L^*(u) = 0$ and $\langle u, e \rangle = 0$. If $u \neq 0$, then by Lemma 11, we can find a $y := \bar{y} - tu \in \operatorname{bdy}(K^*)$. But then, we would have

$y \in \Delta_2$; and by Theorem 17,

$$L^* (y) = L^* (\bar{y}) \overset{*}{\preccurlyeq} 0 = v (\mathcal{G}) \, e^*.$$

In that case—again by Theorem 17—the pair $(\bar{x}, y)$ would also be optimal. However; $y \overset{*}{\not\succ} 0$, contrary to assumption. Therefore we must have $u = 0$. $\square$

Proof of (ii). Note that $\ker (L^*) \cap e^\perp$ is an intersection of subspaces, the former of which is nontrivial, and the latter of which has dimension $\dim (V) - 1$. If $\ker (L^*)$ had dimension greater than one, then it would intersect non-trivially with $e^\perp$ because the sum of their dimensions would exceed $\dim (V)$. Therefore, $0 < \dim (\ker (L^*)) \leq 1$ implying that $\dim (\ker (L^*)) = \dim (\ker (L)) = 1$. $\square$

Proof of (iii). We already have $L (\bar{x}) = 0$ from $\bar{y} \overset{*}{\succ} 0$, so begin by taking a nonzero $u \in \ker (L^*)$ such that $u \notin \operatorname{span} (\{\bar{y}\})$. If there is no such u, then we are done. Note that $\langle u, e \rangle \neq 0$ unless $u = 0$ from item (i), so by scaling we assume that $\langle u, e \rangle = 1$.

Apply Lemma 10 to obtain $y := (1 + t) \, \bar{y} - tu \in \operatorname{bdy} (K^*)$. Since $y \in \Delta_2$, this gives us a second optimal pair $(\bar{x}, y)$ with $y \overset{*}{\not\succ} 0$, contrary to our assumptions. Therefore there is a contradiction inherent in the ability to choose $u \notin \operatorname{span} (\{\bar{y}\})$, and we conclude that $\ker (L^*) = \operatorname{span} (\{\bar{y}\})$. $\square$

Proof of (iv). We already know that $L (\bar{x}) = 0$, and that $\dim (\ker (L)) = 1$. Therefore, $\ker (L) = \operatorname{span} (\{\bar{x}\})$. From this it follows that

$$L^* (V)^\perp = \ker (L) = \operatorname{span} (\{\bar{x}\}) \iff L^* (V) = \bar{x}^\perp.$$

If $\bar{x} \succ 0$, then $\langle \bar{x}, y \rangle > 0$ for any nonzero $y \in K^*$ and thus $K^* \cap \bar{x}^\perp = \{0\}$. But, if we assume on the contrary that $\bar{x} \in \operatorname{bdy} (K)$, then we can find a nonzero $c \in \operatorname{bdy} (K^*)$

with $\langle \bar{x}, c \rangle = 0$ using Proposition 5. Furthermore, since $L^*(V) = \bar{x}^{\perp}$, we have $c = L^*(d)$ for some nonzero $d \in V$.

Now, depending on whether or not $\langle d, e \rangle = 0$, we can apply either Lemma 10 or Lemma 11 (through a similar procedure as in the previous items) to find another $y \in \Delta_2 \cap \operatorname{bdy}(K^*)$ with $L^*(y) \in \operatorname{span}(\{c\})$. For example, when $\langle d, e \rangle = 0$, we will have $y \coloneqq \bar{y} - sd$ where $s < 0$. But this $y \in \operatorname{bdy}(K^*)$ is also optimal, since

$$\langle L(\bar{x}), y \rangle = \langle \bar{x}, L^*(y) \rangle = \langle \bar{x}, \alpha c \rangle = 0.$$

This is a contradiction, so our assumption that $\bar{x} \in \operatorname{bdy}(K)$ was flawed, and $\bar{x} \succ 0$. For uniqueness, recall that $L(x) = 0 = L^*(y)$ for any optimal (x, y). Since $\dim(\ker(L)) = \dim(\ker(L^*)) = 1$, any other optimal pair would comprise multiples of $\bar{x}$ and $\bar{y}$, but only $1\bar{x}$ and $1\bar{y}$ belong to Δ_1 and Δ_2. $\square$

Corollary 14. *A game is completely-mixed if and only if $\bar{y} \overset{*}{\succ} 0$ for each of its optimal pairs $(\bar{x}, \bar{y})$.*

Proof. Use Corollary 12 if necessary, and then apply Theorem 19. $\square$

Completely-mixed games admit nice characterizations. The completely-mixed property is preserved under duality. Moreover, the values of the game $\mathcal{G}$ and its dual $\mathcal{G}^*$ coincide and can be computed explicitly.

Theorem 20. *If $\mathcal{G}$ is completely-mixed, then so is $\mathcal{G}^*$ and $v(\mathcal{G}^*) = v(\mathcal{G})$.*

Proof. Let $\mathcal{G}$ be completely-mixed. Suppose that $(\bar{z}, \bar{w})$ is optimal for the game $\mathcal{G}^*$. We claim that $(\bar{w}, \bar{z})$ is optimal for the game $\mathcal{G}$. From Theorem 17 we have

$L^*(\bar{z}) \overset{*}{\succcurlyeq} v(\mathcal{G}^*)\, e^*$ and that implies,

$$\langle L^*(\bar{z}), \bar{x}\rangle \geq v(\mathcal{G}^*)\langle e^*, \bar{x}\rangle \iff \langle \bar{z}, L(\bar{x})\rangle \geq v(\mathcal{G}^*)\,.$$

Now $(\bar{x}, \bar{y})$ is optimal for the completely-mixed $\mathcal{G}$, so $L(\bar{x}) = v(\mathcal{G})\, e$ by Theorem 18. Substitute to find $v(\mathcal{G}) \geq v(\mathcal{G}^*)$. For the reverse inequality $v(\mathcal{G}^*) \geq v(\mathcal{G})$, apply a similar technique to

$$(L^*)^*(\bar{w}) \preccurlyeq_{(K^*)^*} v(\mathcal{G}^*)\, e \iff L(\bar{w}) \preccurlyeq v(\mathcal{G}^*)\, e.$$

Next we show that $\mathcal{G}^*$ is completely-mixed. Suppose that $(\bar{z}, \bar{w})$ is optimal for the game $\mathcal{G}^*$. From Theorem 17, we have $L^*(\bar{z}) - v(\mathcal{G}^*)\, e^* \overset{*}{\succcurlyeq} 0$. Since $\bar{x} \succ 0$, this gives $\langle L^*(\bar{z}) - v(\mathcal{G}^*)\, e^*, \bar{x}\rangle > 0$ unless $L^*(\bar{z}) - v(\mathcal{G}^*)\, e^* = 0$. Using the fact that $v(\mathcal{G}^*) = v(\mathcal{G})$ and $\bar{y} \overset{*}{\succ} 0 \implies L(\bar{x}) = v(\mathcal{G})\, e$ from Theorem 18,

$$\langle L^*(\bar{z}) - v(\mathcal{G}^*)\, e^*, \bar{x}\rangle = \langle \bar{z}, v(\mathcal{G})\, e\rangle - v(\mathcal{G}^*)\langle e^*, \bar{x}\rangle = 0.$$

So, $L^*(\bar{z}) = v(\mathcal{G}^*)\, e^*$, and by a similar argument, $L(\bar{w}) = v(\mathcal{G}^*)\, e$. Apply Theorem 17 to those two equalities to conclude that $(\bar{w}, \bar{z})$ is optimal for the game $\mathcal{G}$ with corresponding value $v(\mathcal{G}^*) = v(\mathcal{G})$. As a result, $\mathcal{G}^*$ is completely-mixed. □

Theorem 21. *If $\mathcal{G}$ is completely-mixed, then it has a unique optimal pair $(\bar{x}, \bar{y})$, and $v(\mathcal{G}) \neq 0$ if and only if L is invertible. If L^{-1} exists, then*

$$v(\mathcal{G}) = 1/\left\langle L^{-1}(e), e^*\right\rangle,\ \ \bar{x} = v(\mathcal{G})\, L^{-1}(e)\,,\ \ \textit{and}\ \ \bar{y} = v(\mathcal{G})\, L^{-1}(e^*)\,.$$

Proof. Theorem 19 and Corollary 14 give existence and uniqueness of $(\bar{x}, \bar{y})$. If L^{-1} exists, then $\bar{x}$ and $\bar{y}$ are found by solving $L(\bar{x}) = v(\mathcal{G})\, e$ and $L^*(\bar{y}) = v(\mathcal{G})\, e^*$.

Afterwards, $v(\mathcal{G})$ is found by solving $1 = \langle \bar{x}, e^* \rangle = v(\mathcal{G}) \langle L^{-1}(e), e^* \rangle$. Thus when L^{-1} exists, $v(\mathcal{G}) \neq 0$.

If L is not invertible, then there exists a nonzero $u \in \ker(L)$. Note that $\ker(L) \cap (e^*)^{\perp} = \ker(L + \lambda e \otimes e^*) \cap (e^*)^{\perp}$. Combine Corollary 12 and Theorem 19 to conclude that $\langle u, e^* \rangle \neq 0$. Without loss of generality, let $\langle u, e^* \rangle = 1$. Now $L^*(\bar{y}) = v(\mathcal{G}) e^*$ by Theorem 18, so $\langle L^*(\bar{y}), u \rangle = v(\mathcal{G}) \langle e^*, u \rangle = v(\mathcal{G})$. But, we also have $\langle L^*(\bar{y}), u \rangle = \langle \bar{y}, L(u) \rangle = 0$. Combining these equations gives $v(\mathcal{G}) = 0$. $\square$

All of the results in this section survive if instead we have $\bar{x} \succ 0$ for every optimal pair $(\bar{x}, \bar{y})$. Recall that by Corollary 13, every optimal pair $(\bar{x}, \bar{y})$ for the game $\mathcal{G}$ gives us an optimal pair $(\bar{y}, \bar{x})$ for the game $(-L^*, K^*, e^*, e)$ and vice-versa. So, if every optimal pair for $\mathcal{G}$ has $\bar{x} \succ 0$, then $(-L^*, K^*, e^*, e)$ is completely-mixed. Reasoning backwards, $\mathcal{G}$ is completely-mixed as well.

5.4 Games over Lyapunov-like, Stein-like, and $\mathbf{Z}$-operators

We now turn our attention to games whose payoff operators have some structure. The first class that we investigate are the games $\mathcal{G}$ where $L \in \mathbf{Z}(K)$. One of the reasons why $\mathbf{Z}$-operators are important is for their connection to the stability of dynamical systems—a connection that we will exploit. First we recall some equivalent properties of $\mathbf{Z}$-operators [18].

Definition 34. A linear operator L is *positive stable* if the real part of every eigenvalue of L is positive. A *negative stable* operator is defined similarly.

Theorem 22. *If K is a proper cone in a finite-dimensional real Hilbert space and if $L \in \mathbf{Z}(K)$, then the following are equivalent.*

- L^{-1} *exists,* $L^{-1} \in \pi(K)$*, and* $L^{-1} \in \pi(\operatorname{int}(K))$.
- $L \in \mathbf{S}(K)$.
- L *is positive stable.*
- L^{*} *is positive stable.*
- $L^{*} \in \mathbf{S}(K^{*})$.
- $(L^{*})^{-1}$ *exists,* $(L^{*})^{-1} \in \pi(K^{*})$*, and* $(L^{*})^{-1} \in \pi(\operatorname{int}(K^{*}))$.

The items involving the adjoint follow because the spectra of L and L^{*} coincide. An immediate corollary of Theorem 22 and Proposition 37 is the following.

Corollary 15. *If $L \in \mathbf{Z}(K)$, then $v(\mathcal{G}) > 0$ if and only if L is positive stable.*

Theorem 23. *If $v(\mathcal{G}) > 0$ and if $L \in \mathbf{Z}(K)$, then $\mathcal{G}$ is completely-mixed.*

Proof. Suppose that $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$. Then from Theorem 17 and the fact that $v(\mathcal{G}) > 0$, we have $L(\bar{x}) \succcurlyeq v(\mathcal{G})\, e \succ 0$. Moreover, $L \in \mathbf{S}(K)$ by Proposition 37, so Theorem 22 shows that L^{-1} exists with $L^{-1} \in \pi(K) \cap \pi(\operatorname{int}(K))$. We can thus invert to obtain $\bar{x} \succcurlyeq v(\mathcal{G})\, L^{-1}(e) \succ 0$, showing that $\bar{x} \succ 0$ for any optimal pair $(\bar{x}, \bar{y})$. By Corollary 14, the game $\mathcal{G}$ is completely-mixed. $\square$

Theorem 24. *If L is Lyapunov-like on K in $\mathcal{G}$, then $v(\mathcal{G}^{*}) = v(\mathcal{G})$ and*

(i) $v(\mathcal{G}) > 0$ *if and only if L is positive stable;*

(ii) $v(\mathcal{G}) < 0$ *if and only if* L *is negative stable;*

(iii) if $v(\mathcal{G}) \neq 0$*, then* $\mathcal{G}$ *is completely-mixed.*

Proof. The enumerated claims follow from applying Corollary 15 and Theorem 23 to $\pm L \in \mathbf{Z}(K)$. If $v(\mathcal{G}) \neq 0$, then item (iii) and Theorem 20 show that $v(\mathcal{G}^*) = v(\mathcal{G})$. If $v(\mathcal{G}^*) \neq 0$, then $L^* \in \mathbf{LL}(K^*)$, so item (iii) and Theorem 20 show that $v((\mathcal{G}^*)^*) = v(\mathcal{G}) = v(\mathcal{G}^*)$. If neither $v(\mathcal{G}) \neq 0$ nor $v(\mathcal{G}^*) \neq 0$, then $v(\mathcal{G}) = v(\mathcal{G}^*) = 0$. $\square$

The Stein-like operators of Definition 19 are a discrete analogue of Lyapunov-like operators. Stein's theorem [36] characterizes Schur stability.

Definition 35. An operator L is *Schur stable* if all of its eigenvalues λ have $|\lambda| < 1$.

Note that $A \in \mathrm{cl}(\mathrm{Aut}(K))$ implies that $A \in \pi(K)$. Combined with the fact that every Stein-like operator is a $\mathbf{Z}$-operator, this allows us to deduce the following.

Lemma 12. *If* K *is a proper cone in* V *and if* $L \coloneqq \mathrm{id}_V - A$ *is Stein-like on* K*, then* L *is positive stable if and only if* A *is Schur stable.*

Proof. Lemma 1 in Schneider [34] shows that $L \in \mathbf{S}(K)$ if and only if A is Schur stable. Apply Theorem 22 to $L \in \mathbf{Z}(K)$. $\square$

To characterize games over Stein-like operators, we need the following lemma. We ultimately apply it to the game from Corollary 13.

Lemma 13 (Gowda and Sznajder [16], Lemma 2.7)**.** *If* K *is a proper cone and* $A \in \mathrm{cl}(\mathrm{Aut}(K)) \cap \mathbf{S}(K)$*, then* $A \in \mathrm{Aut}(K)$*.*

Corollary 16. *If $L \coloneqq \mathrm{id}_V - A$ is Stein-like on K and if $-L^* \in \mathbf{S}(K^*)$, then $A \in \mathrm{Aut}(K)$.*

Proof. From the definition of $\mathbf{S}(K^*)$, there exists a $d \overset{*}{\succ} 0$ such that $-L^*(d) = A^*(d) - d \overset{*}{\succ} 0$; in other words, $A^*(d) \overset{*}{\succ} d$. Apply Lemma 13 to conclude that $A^* \in \mathrm{Aut}(K^*)$, which is equivalent to $A \in \mathrm{Aut}(K)$. □

Proposition 41. *If $L \coloneqq \mathrm{id}_V - A$ is Stein-like on K, then $-L^* \in \mathbf{S}(K^*)$ if and only if A^{-1} is Schur stable.*

Proof. Suppose $-L^* \in \mathbf{S}(K^*)$ so that $A \in \mathrm{Aut}(K)$ by Corollary 16 and $-L \in \mathbf{S}(K)$. The set $\mathbf{S}(K)$ is closed under precomposition by $\mathrm{Aut}(K)$, since any element of $\mathrm{Aut}(K)$ is a homeomorphism and will send $\mathrm{int}(K)$ to an open set contained in K. Therefore $A^{-1} \circ (-L) = \mathrm{id}_V - A^{-1} \in \mathbf{S}(K)$. But $\mathrm{id}_V - A^{-1}$ is also a $\mathbf{Z}$-operator on K, so A^{-1} is Schur stable by Theorem 22 and Lemma 12.

To go backwards, note that $\mathrm{id}_V - A^{-1}$ is Stein-like on K, so if A^{-1} is Schur stable, then $\mathrm{id}_V - A^{-1} \in \mathbf{Z}(K) \cap \mathbf{S}(K)$ by Theorem 22 and Lemma 12. Precompose with $A \in \mathrm{Aut}(K)$ to find $-L = A - \mathrm{id}_V \in \mathbf{S}(K)$ implying $-L^* \in \mathbf{S}(K^*)$. □

Theorem 25. *If $L \coloneqq \mathrm{id}_V - A$ is Stein-like on K in $\mathcal{G}$, then $v(\mathcal{G}^*) = v(\mathcal{G})$ and,*

(i) $v(\mathcal{G}) > 0$ if and only if A is Schur stable;

(ii) $v(\mathcal{G}) < 0$ if and only if $A \in \mathrm{Aut}(K)$ and A^{-1} is Schur stable;

(iii) if $v(\mathcal{G}) \neq 0$, then $\mathcal{G}$ is completely mixed.

Proof. The first enumerated item follows from Corollary 15 and Lemma 12. For the second item, note that $v(\mathcal{G}) < 0$ if and only if $-L^* \in \mathbf{S}(K^*)$ by Corollary 13 and Proposition 37. Then apply Proposition 41.

The $v(\mathcal{G}) > 0$ case in the third item is supplied by Theorem 23, so we can assume that $v(\mathcal{G}) < 0$. In that case, item (ii) shows that $A^{-1} \in \operatorname{Aut}(K) \iff (A^*)^{-1} \in \operatorname{Aut}(K^*) \subseteq \pi(K^*)$. If $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$, then from Theorem 17,

$$L^*(\bar{y}) = \bar{y} - A^*(\bar{y}) \overset{*}{\preccurlyeq} v(\mathcal{G})\, e^*.$$

Apply $(A^*)^{-1}$ to both sides and rearrange to obtain

$$\bar{y} \overset{*}{\succcurlyeq} (A^*)^{-1}(\bar{y}) - v(\mathcal{G})\,(A^*)^{-1}(e^*) \overset{*}{\succcurlyeq} -v(\mathcal{G})\,(A^*)^{-1}(e^*)\,.$$

The expression on the right is a positive multiple of an interior point of K^*. As a result, $\bar{y} \overset{*}{\succ} 0$. Since $(\bar{x}, \bar{y})$ was arbitrary, the game $\mathcal{G}$ is completely-mixed.

The proof that $v(\mathcal{G}^*) = v(\mathcal{G})$ is now identical to the one in Theorem 24. □

5.5 Solution by conic programming

Recall the two-person zero-sum matrix game of Example 26. As early as 1947, von Neumann observed [9] that such a game could be expressed as a linear program. The reduction of a two-person zero-sum matrix game to a linear program is straightforward, and its constraints are reminiscient of Theorem 17. Using a similar procedure, we will construct a *conic program* equivalent to a linear game.

Definition 36. If K_1 and K_2 are closed convex cones in a finite-dimensional real Hilbert space W and if $b, c \in W$, then the primal $\mathfrak{P}$ and dual $\mathfrak{D}$ conic programming problems are,

$$\mathfrak{P} \coloneqq \begin{cases} \text{minimize} & \langle b, \xi \rangle \\ \text{subject to} & \Lambda\left(\xi\right) \succcurlyeq_{K_2} c \\ & \xi \succcurlyeq_{K_1} 0 \end{cases} \qquad \mathfrak{D} \coloneqq \begin{cases} \text{maximize} & \langle c, \gamma \rangle \\ \text{subject to} & \Lambda^*\left(\gamma\right) \preccurlyeq_{K_1^*} b \\ & \gamma \succcurlyeq_{K_2^*} 0. \end{cases}$$

A *primal feasible* point satisfies the cone constraints of $\mathfrak{P}$. A *primal optimal solution* is a primal feasible $\bar{\xi}$ such that $\left\langle b, \bar{\xi} \right\rangle \leq \langle b, \xi \rangle$ for all primal feasible ξ. The *value* of a primal feasible point ξ is $\langle b, \xi \rangle$. A *dual feasible* point, its value, and *dual optimal solution* are defined similarly.

The strong duality of linear programming does not necessarily hold for conic programs. Nevertheless, any primal feasible ξ provides an upper bound $\langle b, \xi \rangle$ on the value of any dual optimal solution. Likewise, any dual feasible γ provides a lower bound $\langle c, \gamma \rangle$ on the value of any primal optimal solution. Therefore, if we can find a primal/dual feasible pair (ξ, γ) such that $\langle b, \xi \rangle = \langle c, \gamma \rangle$, then we know that (ξ, γ) are primal/dual optimal solutions to $\mathfrak{P}$ and $\mathfrak{D}$ in Definition 36.

Recall from Theorem 17 that any optimal pair $(\bar{x}, \bar{y})$ for the game $\mathcal{G}$ satisfies $L\left(x\right) \succcurlyeq \nu e$ and $L^*\left(\bar{y}\right) \overset{*}{\preccurlyeq} \nu e^*$. It therefore suffices to consider pairs meeting those criteria. If we express those criteria along with the constraints on Δ_1 and Δ_2, then we arrive at the following descriptions of the players' objectives,

$$\mathtt{p1} := \begin{cases} \text{maximize} & \nu \\ \text{subject to} & x \in K \\ & \langle x, e^* \rangle = 1 \\ & \nu \in \mathbb{R} \\ & L(x) \succcurlyeq \nu e \end{cases} \qquad \mathtt{p2} := \begin{cases} \text{minimize} & \omega \\ \text{subject to} & y \in K^* \\ & \langle y, e \rangle = 1 \\ & \omega \in \mathbb{R} \\ & L^*(y) \overset{*}{\preccurlyeq} \omega e^*. \end{cases}$$

We know that an optimal pair exists and that $\omega = \nu = v(\mathcal{G})$ for that pair. Moreover, the optimization problems $\mathtt{p1}$ and $\mathtt{p2}$ can be converted into primal/dual conic programs satisfying Definition 36.

Theorem 26. *In the game $\mathcal{G}$, there exists a pair of dual conic programs $(\mathfrak{P}, \mathfrak{D})$ such that every optimal pair of $\mathcal{G}$ induces a pair of primal/dual optimal solutions to $(\mathfrak{P}, \mathfrak{D})$, and every pair of primal/dual optimal solutions to $(\mathfrak{P}, \mathfrak{D})$ contains an optimal pair of $\mathcal{G}$.*

Proof. Define the (real) product space $W := V \times \mathbb{R} \times \mathbb{R}$ in which $\mathfrak{P}$ and $\mathfrak{D}$ will take place. In this product space, our free variables are $\xi := (x, \nu_1, \nu_2)^T \in W$ and $\gamma := (y, \omega_1, \omega_2)^T \in W$. The remaining data are,

$$K_2 := K_1 := K \times \mathbb{R}_+ \times \mathbb{R}_+ \subseteq W,$$

$$c := b := (0, -1, 1)^T \in W, \text{ and}$$

$$\Lambda := \begin{bmatrix} L & -e & e \\ -\langle \cdot, e^* \rangle & 0 & 0 \\ \langle \cdot, e^* \rangle & 0 & 0 \end{bmatrix} \in \mathcal{B}(W).$$

As a result, $K_2^* = K_1^* = K^* \times \mathbb{R}_+ \times \mathbb{R}_+$, and $\Lambda^* = \Lambda^T$ can be found by blocktransposition. Substitute these data into Definition 36 to obtain

$$\mathfrak{P} = \begin{cases} \text{minimize} & \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ \nu_1 \\ \nu_2 \end{bmatrix} \right\rangle \\ \text{subject to} & \begin{bmatrix} L & -e & e \\ -\langle \cdot, e^* \rangle & 0 & 0 \\ \langle \cdot, e^* \rangle & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \nu_1 \\ \nu_2 \end{bmatrix} \succcurlyeq_{K_1} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ & \begin{bmatrix} x \\ \nu_1 \\ \nu_2 \end{bmatrix} \succcurlyeq_{K_1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

and

$$\mathfrak{D} = \begin{cases} \text{maximize} & \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} y \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\rangle \\ \text{subject to} & \begin{bmatrix} L^* & -e^* & e^* \\ -\langle \cdot, e \rangle & 0 & 0 \\ \langle \cdot, e \rangle & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \omega_1 \\ \omega_2 \end{bmatrix} \preccurlyeq_{K_1^*} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ & \begin{bmatrix} y \\ \omega_1 \\ \omega_2 \end{bmatrix} \succcurlyeq_{K_1^*} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{cases}$$

Simplify and note that maximizing any real function $f(x)$ is equivalent to minimizing $-f(x)$. Define $\nu := \nu_1 - \nu_2$ and $\omega := \omega_1 - \omega_2$. Then,

$$\mathfrak{P} = \begin{cases} \text{maximize} & \nu_1 - \nu_2 = \nu \\ \text{subject to} & x \in K \\ & \nu_1 \in \mathbb{R}_+ \\ & \nu_2 \in \mathbb{R}_+ \\ & \langle x, e^* \rangle = 1 \\ & L(x) \succcurlyeq \nu e \end{cases} \qquad \mathfrak{D} = \begin{cases} \text{minimize} & \omega_1 - \omega_2 = \omega \\ \text{subject to} & y \in K^* \\ & \omega_1 \in \mathbb{R}_+ \\ & \omega_2 \in \mathbb{R}_+ \\ & \langle y, e \rangle = 1 \\ & L^*(y) \overset{*}{\preccurlyeq} \omega e^*. \end{cases}$$

Suppose that $(\bar{x}, \bar{y})$ is optimal for the game $\mathcal{G}$. Then $\bar{x} \in K$ with $\langle \bar{x}, e^* \rangle = 1$, and $L(x) \succcurlyeq v(\mathcal{G})\, e$ from Theorem 17. If $v(\mathcal{G}) \geq 0$, set $\nu_1 = v(\mathcal{G})$ and $\nu_2 = 0$;

otherwise, set $\nu_1 = 0$ and $\nu_2 = -v(\mathcal{G})$. The point $\xi \coloneqq (\bar{x}, \nu_1, \nu_2)^T$ is now primal feasible with objective value $\nu = v(\mathcal{G})$. Likewise, $\gamma \coloneqq (\bar{y}, \omega_1, \omega_2)^T$ will be dual feasible for suitable ω_1, ω_2 and have objective value $\omega = v(\mathcal{G})$. Since $\nu = \omega = v(\mathcal{G})$, the pair (ξ, γ) contains primal/dual optimal solutions to $(\mathfrak{P}, \mathfrak{D})$.

Now suppose that $\bar{\xi} = (\bar{x}, \bar{\nu}_1, \bar{\nu}_2)^T$ is primal optimal and $\bar{\gamma} = (\bar{y}, \bar{\omega}_1, \bar{\omega}_2)^T$ is dual optimal for $(\mathfrak{P}, \mathfrak{D})$. Through circular reasoning, we know that $\left\langle b, \bar{\xi} \right\rangle = \langle c, \bar{\gamma} \rangle$; otherwise, $\left(\bar{\xi}, \bar{\gamma}\right)$ would be worse than any pair of primal/dual optimal solutions obtainable from $\mathcal{G}$. Thus we can define $\nu \coloneqq \left\langle b, \bar{\xi} \right\rangle = \bar{\nu}_1 - \bar{\nu}_2 = \bar{\omega}_1 - \bar{\omega}_2$. The feasibility of $\bar{\xi}$ and $\bar{\gamma}$ implies that $\bar{x} \in \Delta_1$ with $L(x) \succcurlyeq \nu e$ and $\bar{y} \in \Delta_2$ with $L^*(y) \overset{*}{\preccurlyeq} \nu e^*$. By Theorem 17, the pair $(\bar{x}, \bar{y})$ is optimal for $\mathcal{G}$, and $v(\mathcal{G}) = \nu$. □

Symmetric cones are a special case of self-dual cones. Games over self-dual cones were studied by Gowda and Ravindran [15] and form the basis for this chapter. With regard to Theorem 26, symmetric cones have a nice property.

Corollary 17. *Every linear game over a symmetric cone is solved by an associated symmetric conic program.*

Proof. If K is symmetric, then $K_2 = K_1 = K \times \mathbb{R}_+ \times \mathbb{R}_+$ is symmetric in our construction of the pair $(\mathfrak{P}, \mathfrak{D})$. □

This connection is useful because efficient algorithms to solve symmetric conic programs are known in many cases [2].

Bibliography

[1] C. D. Aliprantis and O. Burkinshaw. *Positive Operators*. Springer, Dordrecht, The Netherlands, 2006.

[2] M. Andersen, J. Dahl, Z. Liu, and L. Vandenberghe. Interior-point methods for large-scale cone programming. In S. Sra, S. Nowozin, and S. J. Wright, editors, *Optimization for machine learning*, Neural Information Processing, chapter 3, pages 55–83. The MIT Press, Cambridge, Massachusetts, 2012.

[3] A. Baker. *Matrix Groups: an introduction to Lie group theory*. Springer undergraduate mathematics series. Springer-Verlag, London, 2002.

[4] A. Barvinok. *A Course in Convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2002.

[5] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, 2009.

[7] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. SIAM, Philadelphia, 2009.

[8] T. Damm. Positive groups on $\mathcal{H}^n$ are completely positive. *Linear Algebra and its Applications*, 393:127–137, 2004.

[9] G. B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, 1963.

[10] L. Elsner. Quasimonotonie und ungleichungen in halbgeordneten räumen. *Linear Algebra and its Applications*, 8:249–261, 1974.

[11] W. Fenchel. *Convex Cones, Sets and Functions*. Princeton University Notes, Princeton University, Princeton, 1953.

[12] F. R. Gantmacher. *The Theory of Matrices*, volume II. AMS Chelsea, Providence, Rhode Island, 2000.

[13] M. S. Gowda. Completely mixed linear games on a self-dual cone. *Linear Algebra and its Applications*, 498:219–230, 2016.

[14] M. S. Gowda and T. Parthasarathy. Complementarity forms of theorems of Lyapunov and Stein, and related results. *Linear Algebra and its Applications*, 320:131–144, 2000.

[15] M. S. Gowda and G. Ravindran. On the game-theoretic value of a linear transformation relative to a self-dual cone. *Linear Algebra and its Applications*, 469:440–463, 2015.

[16] M. S. Gowda and R. Sznajder. On the irreducibility, self-duality, and non-homogeneity of completely positive cones. *Electronic Journal of Linear Algebra*, 26:177–191, 2013.

[17] M. S. Gowda, R. Sznajder, and J. Tao. The automorphism group of a completely positive cone and its Lie algebra. *Linear Algebra and its Applications*, 438:3862–3871, 2013.

[18] M. S. Gowda and J. Tao. Z-transformations on proper and symmetric cones. *Mathematical Programming*, 117:195–222, 2009.

[19] M. S. Gowda and J. Tao. On the bilinearity rank of a proper cone and Lyapunov-like transformations. *Mathematical Programming*, 147:155–170, 2014.

[20] M. S. Gowda and D. Trott. On the irreducibility, Lyapunov rank, and automorphisms of special Bishop-Phelps cones. *Journal of Mathematical Analysis and Applications*, 419:172–184, 2014.

[21] J. Hilgert, K. H. Hofmann, and J. D. Lawson. *Lie groups, convex cones and semigroups*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1989.

[22] I. Kaplansky. A contribution to von Neumann's theory of games. *Annals of Mathematics*, 46(3):474–479, 1945.

[23] S. Karlin. *Mathematical Methods and Theory in Games, Programming, and Economics*. Addison-Wesley Publishing Company, Reading, Massachusetts, 1959.

[24] S. Karlin and W. J. Studden. *Tchebycheff Systems with Applications in Analysis and Statistics*, volume XV of *Pure and Applied Mathematics*. Wiley Interscience Publishers, 1966.

[25] M. G. Krein and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *American Mathematical Society Translations*, 10:199–325, 1962.

[26] J. B. Lasserre. *Moments, Positive Polynomials, and their Applications*. Imperial College Press, London, 2010.

[27] M. Orlitzky. The Lyapunov rank of an improper cone. *Optimization Methods and Software*, 32(1):109–125, 2017.

[28] M. Orlitzky and M. S. Gowda. An improved bound for the Lyapunov rank of a proper cone. *Optimization Letters*, 10:11–17, 2016.

[29] T. Parthasarathy and T. Raghavan. *Some topics in two-person games*. Elsevier, New York, 1971.

[30] T. Raghavan. Completely mixed games and M-matrices. *Linear Algebra and its Applications*, 21:35–45, 1978.

[31] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[32] S. Roman. *Advanced Linear Algebra*, volume 135 of *Graduate Texts in Mathematics*. Springer Science+Business Media, New York, third edition, 2008.

[33] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh. Bilinear optimality constraints for the cone of positive polynomials. *Mathematical Programming*, 129:5–31, 2011.

[34] H. Schneider. Positive operators and an inertia theorem. *Numerische Mathematik*, 7:11–17, 1965.

[35] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM Journal on Numerical Analysis*, 7:508–519, 1970.

[36] P. Stein. Some general theorems on iterants. *Journal of Research of the National Bureau of Standards*, 48:82–32, 1952.

[37] R. J. Stern and H. Wolkowicz. Exponential nonnegativity on the ice cream cone. *SIAM Journal on Matrix Analysis and Applications*, 12:160–165, 1991.

[38] B.-S. Tam. Some results of polyhedral cones and simplicial cones. *Linear and Multilinear Algebra*, 4(4):281–284, 1977.

[39] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2017.

[40] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.