When a maximal angle among cones is nonobtuse

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Abstract

Principal angles between linear subspaces have been studied for their application to statistics, numerical linear algebra, and other areas. In 2005, Iusem and Seeger defined critical angles within a single convex cone as an extension of antipodality in a compact set. Then, in 2016, Seeger and Sossa extended that notion to two cones. This was motivated in part by an application to regression analysis, but also allows their cone theory to encompass linear subspaces which are themselves convex cones.

One obstacle to computing the maximal critical angle between cones is that, in general, the maximum won't occur at generators of the cones. We show that in the special case where the maximal angle between the cones is nonobtuse, it does suffice to check only the generators. This extends some results of Iusem and Seeger, and we show that the special case can be checked at essentially no extra cost.

Finally, we point out how large eigenspaces become problematic during a search for critical angles. We prove a few results that aid in this search, and then focus on computing the maximal angle where we can rule out the nonobtuse case and avoid the associated class of hard problems.

1 Background

The idea of a principal angle between linear subspaces goes back at least to Afriat [1], who set out to construct a linear-algebraic framework that would encompass multivariate statistical analysis. A newer reference whose terminology is closer to our own is Miao and Ben-Israel [18]. The first principal angle between a pair of linear subspaces P and Q is defined by

$$\cos(\theta_1) \coloneqq \max\left(\left\{\langle u, v\rangle \mid u \in P,\ v \in Q,\ \text{and}\ \|u\| = \|v\| = 1\right\}\right), \tag{i}$$

and since the cosine function is decreasing on $[0, \pi]$, we can think of θ_1 as being a minimal angle between the spaces P and Q. The pair (u_1, v_1) achieving the minimal angle θ_1 is called the first pair of principal vectors, and the second principal angle θ_2 can then be defined by considering only those $u \in P$ and $v \in Q$ that are orthogonal to u_1 and v_1 respectively. Subsequent (third, fourth, et cetera) principal angles are defined similarly.

To motivate these definitions, let $P \coloneqq \operatorname{span}(\{p\})$ be the line generated by some unit vector p, and let Q be a subspace of the same ambient space. The maximization problem (i) is then equivalent to

$$\text{minimize } -\langle \alpha p, v\rangle \text{ subject to } \alpha \in \{-1, 1\}, \ v \in Q, \text{ and } \|v\| = 1.$$

This is apparently two least-squares problems, since

$$\|\alpha p - v\|^2 = \langle \alpha p - v, \alpha p - v\rangle = 2 - 2\langle \alpha p, v\rangle.$$

One might therefore think of principal angles and vectors as generalizing least-squares. It would then not be surprising that Björck and Golub [3] were able to use the QR-factorization to compute them.

The first generalization of principal angles to convex cones appears to be Obert [19], who considers two specific cones with applications to differential equations. However, our story begins in earnest with Iusem and Seeger [12]. Two points u and v in a compact set K are *antipodal* if they are "diametrically opposite," which means that $\|u - v\|$ achieves the *diameter of the set* K,

$$\operatorname{diam}(K) \coloneqq \max(\{\|u - v\| \mid u, v \in K\}).$$

The diameter of a closed convex cone is undefined, because cones are unbounded. However, if we consider only its unit-norm elements, then those form a compact set and we can formulate the problem

$$\text{maximize } \|u - v\| \text{ subject to } u, v \in K \text{ and } \|u\| = \|v\| = 1.$$

Maximizing $\|u - v\|^2$ is equivalent to minimizing $\langle u, v\rangle$, which in turn is equivalent to maximizing an angle. This leads Iusem and Seeger to define the maximal angle of a closed convex cone K as

$$\theta_{\max}(K) \coloneqq \sup(\{\arccos(\langle u, v\rangle) \mid u, v \in K \text{ and } \|u\| = \|v\| = 1\}), \tag{ii}$$

and to note that

$$\arccos(\langle u, v\rangle) = \theta_{\max}(K) \iff \|u - v\| = \operatorname{diam}(\{x \in K \mid \|x\| = 1\}).$$

As practical motivation, the number $\theta_{\max}(K)$ appears in Peña and Renegar [20] which is concerned with the distance between some given linear-conic constraints and the set of ill-posed constraints. The authors reformulate these constraints so that interior-point methods can be applied to determine feasibility, and $\theta_{\max}(K)$ appears in some estimates of the backwards-stability of that process. Another application of $\theta_{\max}(K)$ is in determining "how pointed" a given cone K is. Iusem and Seeger call this an *index of pointedness* [11], and it becomes important whenever pointedness of the cone K is a necessary property but can vary depending on some parameter. The authors provide a satisfying answer involving $\theta_{\max}(K)$ a bit later [16], and eventually show that $\theta_{\max}$ is Lipschitz continuous on an appropriate space [23].

Iusem and Seeger prove a number of results for polyhedral cones. In particular, they show that polyhedral cones whose maximal angle is nonobtuse will have an antipodal pair of generators.

Proposition I (Iusem and Seeger [12], Proposition 6.2)**.** *Let K be a polyhedral convex cone with unit-norm generating set G. If $\theta_{max}(K) \leq \pi/2$ and if $g_1, g_2 \in G$ form the largest angle within G, then (g_1, g_2) is an antipodal pair of K.*

In other words: if the maximal angle of a polyhedral cone is known to be nonobtuse, then it will occur at a pair of generators, and those are essentially finite in number. The proof of their proposition involves constructing the Gramian matrix $M = \left[\langle g_i, g_j \rangle\right]$, and therefore relies on the fact that G is finite or (equivalently) that K is polyhedral.

Due to the nonconvexity in (ii), there is a necessary but insufficient condition for a pair $(u, v) \in K^2$ to consist of antipodal points of K. Pairs satisfying the necessary condition are called *critical pairs*, and the angles they form *critical angles*. The set of all critical angles within a cone is its *angular spectrum*, and it provides useful information about the geometric structure of the cone. For example, polyhedral cones have finite angular spectra [17]. Iusem and Seeger explicitly construct cones having infinite angular spectra [14], and conjecture that the angular spectrum of any closed convex cone is nowhere-dense. They also point out an application of critical angles to a complementarity problem involving eigenvalues of a matrix relative to a closed convex cone.

In what the authors consider the third part of a triptych [15], some remaining questions about the angular spectrum are answered. A weaker notion of antipodal pair called a *Nash angular equilibrium* is introduced that satisfies the following relationship: all antipodal pairs are Nash angular equilibria, and all Nash angular equilibria are critical pairs.

Proposition II (Iusem and Seeger [15], Proposition 2)**.** *Let K be a closed convex cone such that $K \neq \{0\}$. If $\langle x, y \rangle > 0$ for all $x, y \in K \setminus \{0\}$, then every Nash angular equilibrium is a pair of normalized extreme rays of K.*

Since antipodal pairs are Nash angular equilibria, Propositions I and II are related. Later we will unify and generalize these two results, allowing for the cones to be non-polyhedral and the inner products to be merely nonnegative.

Iusem and Seeger apply their theory to two classes of cones, ellipsoidal and spectral cones. Spectral cones are particularly important in optimization because they allow one to transfer a problem from a difficult space, such as the space of n-by-n real symmetric matrices, to an easier one like $\mathbb{R}^n$. The authors show that the maximal angle problem is susceptible to this type of attack [13]. These sorts of problems are still an active area of research. For example, the commutation principle used by Iusem and Seeger was later shown by Ramírez, Seeger, and Sossa [21] to hold in a general Euclidean Jordan Algebra, and by Gowda and Jeong [8] to hold in a normal decomposition system.

With respect to practical computation, the next advance is by Gourion and Seeger [6], who devise an algorithm to compute the angular spectrum of a polyhedral cone. This algorithm provides evidence for the cardinality of a random polyhedral cone's angular spectrum. In a subsequent work [7], the authors formally investigate this expected number of critical angles.

As a final application, recall that an $n \times n$ real matrix A is *copositive* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{R}^n_+$. The set of all copositive matrices forms a closed convex cone in the space of $n \times n$ real symmetric matrices. What is the maximal angle that can be formed by two copositive matrices? Hiriart-Urruty and Seeger [10] conjecture that the answer is $3\pi/4$, but Goldberg and Shaked-Monderer [4] show that the maximal angle tends towards π as n grows large.

Seeger and Sossa are responsible for expanding the theory to two cones [24, 25]. The concepts of critical angle, Nash angle, maximal angle, and the associated pairs all extend in a natural way. And since linear subspaces are themselves closed convex cones, this brings us full-circle to a generalization of the principal angles between linear subspaces.

2 Two cones

The natural setting in which to discuss angles between convex cones is in a real Hilbert space. And for reasons that will become apparent, we would like the unit sphere to be compact. So, throughout this section, V will denote a finite-dimensional real Hilbert space.

Definition 1. A nonempty subset K of V is a *cone* if $\alpha K \subseteq K$ for all $\alpha \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V.

Definition 2. The *conic hull* of a nonempty subset X of V is a convex cone,

$$\operatorname{cone}(X) := \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \geq 0,\ m \in \mathbb{N} \right\}.$$

If $\operatorname{cone}(G) = K$, then G *generates* K and the elements of G are *generators* of K. If a finite set generates K, then K is *polyhedral*.

All polyhedral convex cones are closed [22]. Note that for $G \neq \{0\}$ we have

$$\operatorname{cone}(G) = \operatorname{cone}\left(\left\{ \frac{g}{\|g\|} \;\middle|\; g \in G, g \neq 0 \right\} \right),$$

and that therefore no generality is lost if we insist on unit-norm generators.

Definition 3. If K is a subset of V, then the *dual cone* of K is

$$K^* := \{ y \in V \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K \}.$$

If K is contained in K^*, then K is *subdual*.

We adopt the standard definition of the angle θ between two vectors u and v in a Hilbert space, namely $\|u\| \|v\| \cos(\theta) = \langle u, v \rangle$. For nonzero u and v, we divide and set $\cos(\theta) := \langle u / \|u\|, v / \|v\| \rangle$.

Definition 4. The unit sphere in V is denoted by $\mathbb{S}(V)$.

This leads to a small problem: $K = \{0\}$ satisfies Definition 1, but it contains no unit-norm elements. Thus the angle between a given vector and any element of K is undefined. This can be settled by fiat, but nothing of value is gained and the additional special cases complicate the proofs. We therefore follow Seeger et al. and omit the cone $K = \{0\}$ from consideration. For the same reason, we omit the cone $K = V$; we generally need to take duals and $V^* = \{0\}$.

Definition 5. The set of *admissible cones* in V is

$$\mathcal{C}(V) \coloneqq \{K \mid K \text{ is a closed convex cone in } V \text{ and } K \notin \{\{0\}, V\}\}.$$

Reference to the set of admissible cones will simplify the statement of our results. For example, if $P, Q \in \mathcal{C}(V)$, then Seeger and Sossa [24] define the maximal angle between P and Q to be

$$\sup\left(\{\arccos\left(\langle u, v\rangle\right) \mid u \in P \cap \mathbb{S}(V),\ v \in Q \cap \mathbb{S}(V)\}\right). \tag{1}$$

Since $P, Q \in \mathcal{C}(V)$, they contain at least one unit-norm element. Problem (1) is therefore maximizing the continuous function $(u, v) \mapsto \arccos\left(\langle u, v\rangle\right)$ over a nonempty compact set. The supremum is thus achieved, and moreover the arccos function is decreasing on $[-1, 1]$. As a result, we can pose the supremum problem (1) as a minimization problem instead.

Definition 6. If $P, Q \in \mathcal{C}(V)$, then the *maximal angle* between P and Q is

$$\Theta(P, Q) \coloneqq \arccos\left(\min\left(\{\langle u, v\rangle \mid u \in P \cap \mathbb{S}(V),\ v \in Q \cap \mathbb{S}(V)\}\right)\right).$$

If $\arccos\left(\langle \bar{u}, \bar{v}\rangle\right) = \Theta(P, Q)$ for some $\bar{u} \in P \cap \mathbb{S}(V)$ and $\bar{v} \in Q \cap \mathbb{S}(V)$, then $(\bar{u}, \bar{v})$ is an *antipodal pair* of (P, Q). We abbreviate $\Theta(K, K)$ as $\Theta(K)$.

One can define the minimal angle in a completely analogous way, but as Seeger and Sossa explain [24], the maximal angle is in many ways nicer to work with.

Definition 6 involves an optimization problem. As is the standard practice, one can use its Karush–Kuhn–Tucker conditions to obtain necessary conditions for optimality. The search space is then reduced to the pairs of points satisfying the necessary conditions. Seeger and Sossa call these *critical pairs* [24].

Definition 7. If $P, Q \in \mathcal{C}(V)$, then (u, v) is a *critical pair* of (P, Q) if

$$\begin{aligned} u &\in P \cap \mathbb{S}(V), \\ v &\in Q \cap \mathbb{S}(V), \\ v - \langle u, v\rangle\, u &\in P^*, \text{ and} \\ u - \langle u, v\rangle\, v &\in Q^*. \end{aligned}$$

In that case, $\arccos\left(\langle u, v\rangle\right)$ is a *critical angle* of (P, Q). The set of all such angles is the *angular spectrum* of (P, Q), and is denoted by $\Gamma(P, Q)$.

One final class of pairs, the Nash angular equilibria, lies conceptually between antipodal pairs and critical pairs. For consistency, we will refer to Nash angular equilibria as simply "Nash pairs."

Definition 8. If $P, Q \in \mathcal{C}(V)$, then $(\bar{u}, \bar{v})$ is a *Nash pair* of (P, Q) if

$$\begin{aligned} \bar{u} &\in P \cap \mathbb{S}(V), \\ \bar{v} &\in Q \cap \mathbb{S}(V), \\ \langle \bar{u}, \bar{v} \rangle &\le \langle u, \bar{v} \rangle \text{ for all } u \in P \cap \mathbb{S}(V), \text{ and} \\ \langle \bar{u}, \bar{v} \rangle &\le \langle \bar{u}, v \rangle \text{ for all } v \in Q \cap \mathbb{S}(V). \end{aligned}$$

In that case, $\arccos(\langle \bar{u}, \bar{v} \rangle)$ is a *Nash angle* of (P, Q).

Seeger and Sossa mention Nash angular equilibria for pairs of cones [25], but their true heritage is the earlier work of Iusem and Seeger [15] for a single cone. The following relationship holds for both pairs of points and for angles:

$$\text{antipodal} \implies \text{Nash} \implies \text{critical}.$$

To remain completely precise in our prose, we record the following.

Definition 9. The angle θ is *nonobtuse* if $\theta \in [0, \pi/2]$.

In the later sections, we work in the real n-space $\mathbb{R}^n$ rather than in a general Hilbert space. The space $\mathbb{R}^n$ has the usual inner product and standard basis $\{e_1, e_2, \ldots, e_n\}$. Linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$ are represented by matrices $A \in \mathbb{R}^{n \times m}$, and the Moore–Penrose pseudoinverse of A is denoted by A^+. The cardinality of a set X is $\operatorname{card}(X)$; so, for example, $\operatorname{card}(\{e_1, e_2, \ldots, e_n\}) = n$. The nonnegative orthant in $\mathbb{R}^n$ is $\mathbb{R}^n_+ := \operatorname{cone}(\{e_1, e_2, \ldots, e_n\})$.

3 The nonobtuse case

Using Definition 6, we are immediately able to characterize when the maximal angle between two closed convex cones is nonobtuse.

Theorem 1. *If $P, Q \in \mathcal{C}(V)$ for a finite-dimensional real Hilbert space V, then the following are equivalent:*

1. $\Theta(P, Q) \in [0, \pi/2]$
2. $P \subseteq Q^*$
3. $Q \subseteq P^*$

Proof. To see that Item 1 implies Item 2, suppose that $\Theta(P, Q) \in [0, \pi/2]$. A minimal pair $(\bar{u}, \bar{v})$ in Definition 6 must have $\langle \bar{u}, \bar{v} \rangle \in [0, 1]$. But then the fact that $\langle u, v \rangle \ge \|u\| \|v\| \langle \bar{u}, \bar{v} \rangle \ge 0$ for all $u \in P$ and $v \in Q$ shows that $P \subseteq Q^*$.

For the converse implication: if $P \subseteq Q^*$, then $\langle u, v \rangle \ge 0$ for all $u \in P$ and $v \in Q$. And by Cauchy–Schwarz, we have $\langle u, v \rangle \le \|u\| \|v\| = 1$ for any feasible pair (u, v) in Definition 6. Thus $\langle u, v \rangle \in [0, 1]$ for any feasible pair, and in particular for the minimal pair $(\bar{u}, \bar{v})$. It follows that $\Theta(P, Q) = \arccos(\langle \bar{u}, \bar{v} \rangle) \in [0, \pi/2]$, and that Items 1 and 2 are equivalent.

To see that Item 2 implies Item 3, start with $P \subseteq Q^*$ and take duals on both sides to find $P^* \supseteq (Q^*)^*$. Use the fact that Q is a closed convex cone and Ben-Israel's [2] Theorem 1.5 to conclude that $(Q^*)^* = Q$. The proof of the converse is identical with the roles of P and Q switched. $\square$

From now on, we will cite the geometric condition $P \subseteq Q^*$ when we wish to convey that the maximal angle between P and Q is nonobtuse. Note that if $P = \operatorname{cone}(G)$ and $Q = \operatorname{cone}(H)$ for two sets G and H, then the condition $P \subseteq Q^*$ is equivalent to saying that $\langle g, h \rangle \geq 0$ for all $g \in G$ and $h \in H$. One implication in that equivalence is trivial, and the other follows immediately from the bilinearity of the inner product using Definition 2. Thus when P and Q are polyhedral, we can actually determine whether or not Theorem 1 applies.

Recall from Definition 3 that a subset K of a Hilbert space V is subdual if $K \subseteq K^*$. If $K \in \mathcal{C}(V)$ is subdual, then we can set $P := K$ and $Q := K$ in Theorem 1 to see that the maximal angle in K is nonobtuse.

Corollary 1. *If $K \in \mathcal{C}(V)$ for a finite-dimensional real Hilbert space V, then $\Theta(K) \in [0, \pi/2]$ if and only if $K \subseteq K^*$.*

Example 1. Consider the completely-positive cone in $\mathbb{R}^n$,

$$\mathbf{CP}\left(\mathbb{R}^n_+\right) := \operatorname{cone}\left(\left\{uu^T \;\middle|\; u \in \mathbb{R}^n_+\right\}\right),$$

which is known to be subdual [9]. Set $K := \mathbf{CP}\left(\mathbb{R}^n_+\right)$ in Corollary 1 to see that $\Theta(K) \in [0, \pi/2]$. Now both $e_1 e_1^T$ and $e_2 e_2^T$ belong to $\mathbf{CP}\left(\mathbb{R}^n_+\right)$, and

$$\left\langle e_1 e_1^T, e_2 e_2^T \right\rangle = \langle e_1, e_2 \rangle^2 = 0.$$

Since $\arccos(0) = \pi/2$ is as large as possible, we have $\Theta\left(\mathbf{CP}\left(\mathbb{R}^n_+\right)\right) = \pi/2$.

While our main focus is the nonobtuse case, there is one obtuse angle whose criticality and maximality are easy to characterize, so we record this fact for later. One implication follows from Seeger and Sossa's Proposition 4.1 and Proposition 4.2 [24], but a direct proof of the entire result is straightforward.

Proposition 1. *If $P, Q \in \mathcal{C}(V)$ for a finite-dimensional real Hilbert space V, then the following are equivalent:*

1. $P \cap -Q \neq \{0\}$
2. $\Theta(P, Q) = \pi$
3. $\pi \in \Gamma(P, Q)$

Intuitively, the maximal angle should not occur "inside" the cones. This is made precise by Seeger and Sossa's Theorem 2.3 [24]; but perhaps surprisingly, the maximal angle need not occur at a pair of generators.

Example 2. Let $P := \operatorname{cone}(\{e_1, e_2, -e_2\})$ be the right half-plane in $\mathbb{R}^2$. If $Q = \operatorname{cone}(\{-e_1 + e_2, -e_1 - e_2\})$, then $e_1 \in P \cap -Q$ and it follows from Proposition 1 that $\Theta(P, Q) = \pi$. However it is visually obvious that no pair of generators forms an angle of π.

If the maximal angle always occurred at a pair of generators, then—at least for polyhedral cones—we could simply check them all. Alas, this is true only if the maximal angle is known to be nonobtuse. We set out to prove the following.

Corollary 2. *Let V be a finite-dimensional real Hilbert space, and let $P, Q \in \mathcal{C}(V)$ be such that $P \subseteq Q^*$. If $P = \operatorname{cone}(G)$ and $Q = \operatorname{cone}(H)$ for $G, H \subseteq \mathbb{S}(V)$, then some $(g, h) \in G \times H$ is an antipodal pair for (P, Q).*

However, it turns out that all Nash angles (not just the largest) will occur at pairs of generators when the maximal angle between the cones is nonobtuse.

Lemma 1. *Let V be a finite-dimensional real Hilbert space, and let $P, Q \in \mathcal{C}(V)$ be such that $P \subseteq Q^*$. If $(\bar{u}, \bar{v})$ is a Nash pair for (P, Q) corresponding to the angle θ and if $\bar{v} \in \operatorname{cone}(\{v_1, v_2\})$ for $v_1, v_2 \in Q \cap \mathbb{S}(V)$, then either $(\bar{u}, v_1)$ or $(\bar{u}, v_2)$ is also a Nash pair corresponding to the angle θ.*

Proof. Let $\bar{v} = \alpha_1 v_1 + \alpha_2 v_2$ with $\alpha_1, \alpha_2 \geq 0$ and note that the triangle inequality gives $1 = \|\alpha_1 v_1 + \alpha_2 v_2\| \leq \alpha_1 + \alpha_2$. Since $P \subseteq Q^*$, both $\langle \bar{u}, v_1 \rangle$ and $\langle \bar{u}, v_2 \rangle$ are nonnegative. Suppose without loss of generality that $\langle \bar{u}, v_1 \rangle \leq \langle \bar{u}, v_2 \rangle$. Then,

$$\langle \bar{u}, \bar{v} \rangle = \alpha_1 \langle \bar{u}, v_1 \rangle + \alpha_2 \langle \bar{u}, v_2 \rangle \geq (\alpha_1 + \alpha_2) \langle \bar{u}, v_1 \rangle \geq \langle \bar{u}, v_1 \rangle .$$

But $(\bar{u}, \bar{v})$ is a Nash pair, so from Definition 8, we also have that $\langle \bar{u}, \bar{v} \rangle \leq \langle \bar{u}, v_1 \rangle$. It follows that $\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}, v_1 \rangle$, and that $(\bar{u}, v_1)$ is a Nash pair. □

Theorem 2. *Let V be a finite-dimensional real Hilbert space, and let $P, Q \in \mathcal{C}(V)$ be such that $P \subseteq Q^*$. If $P = \operatorname{cone}(G)$ and $Q = \operatorname{cone}(H)$ for $G, H \subseteq \mathbb{S}(V)$, then every Nash angle of (P, Q) is achieved by some $(g, h) \in G \times H$.*

Proof. Let $(\bar{u}, \bar{v})$ be a Nash pair of (P, Q). Using Definition 2, write $\bar{v} \in \operatorname{cone}(H)$ as $\bar{v} = \alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_m h_m$ for $h_j \in H$. Applying Lemma 1 at most $m - 1$ times shows that there exists a j for which $(\bar{u}, h_j)$ is also a Nash pair of (P, Q). Now from Theorem 1, deduce that $Q \subseteq P^*$ as well. Play the same game with $\bar{u}$ to conclude that some (g_i, h_j) is a Nash pair for (P, Q). □

Corollary 2 now follows easily given that maximal angles are Nash angles.

Corollary 3. *If $\operatorname{cone}(G) \in \mathcal{C}(V)$ is subdual in a finite-dimensional real Hilbert space V for $G \subseteq \mathbb{S}(V)$, then some $(g_1, g_2) \in G^2$ is an antipodal pair for $\operatorname{cone}(G)$.*

Example 3. We introduced the completely-positive cone in $\mathbb{R}^n$ in Example 1. Now let V be any finite-dimensional real Hilbert space, and let $K \in \mathcal{C}(V)$ be such that $-K \cap K = \{0\}$. We consider the *completely-positive cone of K*,

$$\mathbf{CP}(K) := \operatorname{cone}(\{u \otimes u \mid u \in K \cap \mathbb{S}(V)\}).$$

This cone is known to be subdual [9]. Apply Corollary 3 to conclude that

$$\begin{aligned}\Theta(\mathbf{CP}(K)) &= \arccos(\min(\{\langle u \otimes u, v \otimes v \rangle \mid u, v \in K \cap \mathbb{S}(V)\})) \\ &= \arccos\left(\min\left(\left\{\langle u, v \rangle^2 \,\middle|\, u, v \in K \cap \mathbb{S}(V)\right\}\right)\right).\end{aligned}$$

If $\Theta(K)$ is nonobtuse, then every inner product $\langle u, v\rangle$ above is nonnegative, and minimizing the squares is the same as squaring the minimum. Thus,

$$\Theta(\mathbf{CP}(K)) = \arccos\left([\cos(\Theta(K))]^2\right) \text{ if } \Theta(K) \in [0, \pi/2].$$

On the other hand, if $\Theta(K) > \pi/2$, then there exist $u, v \in K \cap \mathbb{S}(V)$ having $\langle u, v\rangle < 0$. Since $-K \cap K = \{0\}$, the origin is not a convex combination of u and v. But K is a convex cone, so the image of $f : [0, 1] \to V$ defined by

$$f(\alpha) \coloneqq \frac{\alpha v + (1-\alpha)u}{\|\alpha v + (1-\alpha)u\|}$$

is contained in $K \cap \mathbb{S}(V)$. The composite $\alpha \mapsto \langle u, f(\alpha)\rangle$ is continuous, and the intermediate value theorem thus ensures the existence of a $v' \in K \cap \mathbb{S}(V)$ such that $\langle u, v'\rangle = 0$. As in Example 1, this shows that $\Theta(\mathbf{CP}(K)) = \pi/2$. Combining these cases, we see that the maximal angle within $\mathbf{CP}(K)$ is a function of the maximal angle within K itself, namely

$$\Theta(\mathbf{CP}(K)) = \arccos\left([\max(\{0, \cos(\Theta(K))\})]^2\right).$$

4 Finding critical angles

Recall that the maximal angle between P and Q is necessarily a critical angle of (P, Q). This motivates Seeger and Sossa's algorithm, based on their Theorem 7.3 [24], to find critical angles. Later we incorporate the results from the previous section into this algorithm, and ultimately focus on finding the maximal angle. But first, we introduce the notation and recall their theorem.

Definition 10. If $G \coloneqq \{g_1, g_2, \ldots, g_p\}$ is a subset of some vector space, then

$$\mathcal{I}(G) \coloneqq \left\{I \subseteq \{1, 2, \ldots, p\} \;\middle|\; I \neq \emptyset,\ \{g_i\}_{i \in I} \text{ is linearly-independent}\right\}.$$

Thus $\{\{g_i \mid i \in I\} \mid I \in \mathcal{I}(G)\}$ is the set of linearly-independent subsets of G.

Definition 11. If $I \subseteq \mathbb{N}$ is nonempty, then we will write I_k to denote indexing into the set I using the canonical order on $\mathbb{N}$. For example, if $I = \{3, 1, 4\}$, then $I_1 = 1$, $I_2 = 3$, and $I_3 = 4$ because $1 \leq 3 \leq 4$.

To calculate critical angles in practice we will use matrices, and must therefore drop the pretense of working in a general Hilbert space V. So from now on, we will fix $V = \mathbb{R}^n$ with the usual basis and inner product.

Definition 12. Let $G \coloneqq \{g_1, g_2, \ldots, g_p\}$ and $H \coloneqq \{h_1, h_2, \ldots, h_q\}$ be subsets of $\mathbb{R}^n$. If $I \subseteq \{1, 2, \ldots, p\}$ and $J \subseteq \{1, 2, \ldots, q\}$, then we define two matrices

$$G_I \coloneqq \begin{bmatrix} g_{I_1} & g_{I_2} & \cdots & g_{I_{\mathrm{card}(I)}} \end{bmatrix} \in \mathbb{R}^{n \times \mathrm{card}(I)}$$
$$H_J \coloneqq \begin{bmatrix} h_{J_1} & h_{J_2} & \cdots & h_{J_{\mathrm{card}(J)}} \end{bmatrix} \in \mathbb{R}^{n \times \mathrm{card}(J)},$$

whose columns are elements of G and H indexed by I and J respectively.

With that out of the way, we restate their theorem.

Theorem 3 (Seeger and Sossa [24], Theorem 7.3)**.** *Let* $G \coloneqq \{g_1, g_2, \ldots, g_p\}$ *and* $H \coloneqq \{h_1, h_2, \ldots, h_q\}$ *be subsets of* $\mathbb{S}(\mathbb{R}^n)$*. If* $P = \operatorname{cone}(G)$*,* $Q = \operatorname{cone}(H)$*, and* $P, Q \in \mathcal{C}(\mathbb{R}^n)$*, then* θ *is a critical angle of* (P, Q) *if and only if there exist* $I \in \mathcal{I}(G)$*,* $J \in \mathcal{I}(H)$*,* $\xi \in \mathbb{R}^{\operatorname{card}(I)}$*, and* $\eta \in \mathbb{R}^{\operatorname{card}(J)}$ *satisfying*

$$\begin{bmatrix} 0 & G_I^T H_J \\ H_J^T G_I & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \cos(\theta) \begin{bmatrix} G_I^T G_I & 0 \\ 0 & H_J^T H_J \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{2}$$

subject to

$$\begin{aligned} \langle g_k, H_J \eta - \cos(\theta) G_I \xi \rangle &\geq 0 \text{ for all } k \notin I, \\ \langle h_\ell, G_I \xi - \cos(\theta) H_J \eta \rangle &\geq 0 \text{ for all } \ell \notin J, \\ \left\langle \xi, G_I^T G_I \xi \right\rangle &= 1, \\ \left\langle \eta, H_J^T H_J \eta \right\rangle &= 1, \\ \xi &> 0 \text{ componentwise, and} \\ \eta &> 0 \text{ componentwise.} \end{aligned} \tag{3}$$

In that case, the critical angle θ *corresponds to the pair* $(u, v) = (G_I \xi, H_J \eta)$*.*

Based on that result, Seeger and Sossa sketch the following algorithm.

Algorithm 1 Compute the angular spectrum of (P, Q)

Input: Sets G and H that generate $P, Q \in \mathcal{C}(\mathbb{R}^n)$ as in Theorem 3.
Output: The set $\Gamma(P, Q)$ of all critical angles of (P, Q).

```
Γ ← ∅
for all I ∈ 𝓘(G) and J ∈ 𝓘(H) do
    Find all cos(θ), ξ, and η that solve Equation (2)
    if any cos(θ), ξ, and η satisfy (3) then
        Γ ← Γ ∪ {θ}
    end if
end for
return Γ
```

In Section 5, we discuss a difficulty that arises while implementing this algorithm. But before we proceed, we show that Equation (2) can be restated as a standard eigenvalue problem whose size can be cut roughly in half. To streamline the notation, we introduce the Moore–Penrose pseudoinverse.

Definition 13. The *Moore–Penrose pseudoinverse* of $A \in \mathbb{R}^{m \times n}$ is the unique matrix $A^+ \in \mathbb{R}^{n \times m}$ satisfying $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^T = AA^+$, and $(A^+A)^T = A^+A$.

The Moore–Penrose pseudoinverse can be computed efficiently using the singular value decomposition. We will need only one well-known property of the pseudoinverse, stated in Section 5.5.2 of Golub and Van Loan [5].

Proposition 2. *If* $A \in \mathbb{R}^{n \times m}$ *has rank* m*, then* $A^{+} = \left(A^{T}A\right)^{-1}A^{T}$*, and both* AA^{+} *and* $A^{+}A$ *are orthogonal projections.*

Proposition 3. *In Theorem 3, Equation (2) is equivalent to*

$$\begin{bmatrix} 0 & G_I^{+}H_J \\ H_J^{+}G_I & 0 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \cos(\theta)\begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{4}$$

Proof. By construction, the columns of G_I and H_J form linearly-independent sets. The matrices $G_I^{T}G_I$ and $H_J^{T}H_J$ are therefore positive-definite and can be inverted directly to obtain

$$\begin{bmatrix} 0 & \left(G_I^{T}G_I\right)^{-1}G_I^{T}H_J \\ \left(H_J^{T}H_J\right)^{-1}H_J^{T}G_I & 0 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \cos(\theta)\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Now Proposition 2 shows that $\left(G_I^{T}G_I\right)^{-1}G_I^{T} = G_I^{+}$ and likewise for H_J. □

Having expressed Equation (2) as a standard eigenvalue problem, we now consider two cases: when the eigenvalue $\cos(\theta)$ is zero, and when it is not.

Proposition 4. *In the context of Proposition 3, the eigenspace corresponding to the eigenvalue* $\cos(\theta) = 0$ *is* $\ker\left(H_J^{T}G_I\right) \times \ker\left(G_I^{T}H_J\right)$*.*

Proof. Obvious after setting $\cos(\theta) = 0$ in the equivalent Equation (2). □

Proposition 5. *If* $\cos(\theta) \neq 0$ *in the context of Proposition 3, then the following are equivalent for the vectors* $\xi \in \mathbb{R}^{\operatorname{card}(I)}$ *and* $\eta \in \mathbb{R}^{\operatorname{card}(J)}$*:*

1. *The block vector* $(\xi, \eta)^{T}$ *is a solution to Equation (4).*
2. ξ *is a solution to* $G_I^{+}H_JH_J^{+}G_I\xi = (\cos(\theta))^{2}\xi$ *and* $\eta = H_J^{+}G_I\xi/\cos(\theta)$*.*
3. η *is a solution to* $H_J^{+}G_IG_I^{+}H_J\eta = (\cos(\theta))^{2}\eta$ *and* $\xi = G_I^{+}H_J\eta/\cos(\theta)$*.*

Proof. Write Equation (4) as a system of two equations. Divide by $\cos(\theta) \neq 0$ on the right-hand side, and substitute either equation into the other. □

In practice, there could be a subtle difference between the problems in Proposition 5. If Equation (4) is solved for an eigenvector $(\xi, \eta)^{T}$, then for example it admits solutions where $\xi = 0$ and $\eta \neq 0$. On the other hand, if we treat Item 2 as an eigenvalue problem, it admits only solutions where $\xi \neq 0$. But notice that the conditions (3) require both ξ and η to be nonzero. As a result, we will not overlook any feasible solutions $(\xi, \eta)^{T}$ if we choose to solve one of the smaller problems in Items 2 and 3 for eigenvectors.

The formulation of the two smaller problems in Proposition 5 is similar to the remarks following Theorem 8.6 in Iusem and Seeger [12]. Note that the size of the original problem in Equation (4) is $\operatorname{card}(I) + \operatorname{card}(J)$, and that by choosing the smaller of the two problems in Proposition 5 we obtain a problem of size $\min(\{\operatorname{card}(I), \operatorname{card}(J)\})$.

5 The problem with eigenspaces

If all of the eigenspaces that appear as solutions to Equation (4) have dimension one, then Algorithm 1 works more or less as stated because checking the feasibility of (3) is straightforward. Suppose $\left\{(\xi,\eta)^T\right\}$ is a basis for such an eigenspace. If possible, multiply by -1 to make both $\xi,\eta > 0$ componentwise. Then scale by a factor of $1/\left\|G_I\xi\right\|$ so that ξ satisfies $\left\langle \xi, G_I^T G_I \xi\right\rangle = 1$. The scaled vector $\pm(\xi,\eta)^T / \left\|G_I\xi\right\|$ is the only element of the eigenspace that satisfies both the positivity and norm constraints, and it suffices to check the remaining two inequalities in (3). But what if we encounter an eigenspace of higher dimension? This situation occurs in some relatively simple examples.

Example 4. Define $P, Q \coloneqq \mathbb{R}^2_+$ with generators $G, H \coloneqq \{e_1, e_2\}$. If $I = \{1\}$ and $J = \{2\}$, then Equation (2) reduces to

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \cos(\theta) \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

with the obvious two-dimensional eigenspace $\operatorname{span}(\{e_1, e_2\})$ corresponding to $\cos(\theta) = 0$. But note that neither $(\xi,\eta)^T = e_1$ nor $(\xi,\eta)^T = e_2$ satisfy the positivity constraints in (3). We are thus left searching all of $\mathbb{R}^2$ for a solution to the system of constraints. The assignment $(\xi,\eta)^T = (1,1)^T$ works in this case by inspection, but in general this problem will be hard.

Example 5. Suppose $\operatorname{card}(I) = \operatorname{card}(J) = n$ for two cones $P, Q \in \mathcal{C}(\mathbb{R}^n)$ in Theorem 3. By construction, both G_I and H_J are square with linearly-independent columns. Substituting $G_I^+ = G_I^{-1}$ and $H_J^+ = H_J^{-1}$ into Equation (4) and multiplying, we obtain the system

$$\begin{aligned} H_J\eta &= \cos(\theta)\, G_I\xi \\ G_I\xi &= \cos(\theta)\, H_J\eta. \end{aligned}$$

It should be clear that for either $\cos(\theta) = 1$ or $\cos(\theta) = -1$, any choice of $\xi, \eta \in \mathbb{R}^n$ will satisfy the system.

At present it is not clear how to adapt Algorithm 1 to work around this difficulty. Since the Nash angles are in general a proper subset of the critical angles, Theorem 2 doesn't apply directly to Algorithm 1. However, checking whether or not $P \subseteq Q^*$ involves computing the inner products between the generators of P and Q. Then, if it turns out that $P \subseteq Q^*$, Corollary 2 says that we will have computed $\Theta(P,Q)$ as a side effect of that process. This has two benefits. First, $\Theta(P,Q)$ is itself a critical angle, and can be recorded as such. Second, it allows us to ignore any eigenvalues (and their eigenspaces) that are less than or equal to $\cos(\Theta(P,Q))$, because they correspond to angles greater than or equal to the maximal angle. However, it may be the case that $P \nsubseteq Q^*$, and we don't want to waste too much time checking a special case. Fortunately, as the next proposition shows, these benefits come cheaply. This will be even more important in Section 6, where Theorem 2 becomes invaluable.

Proposition 6. *In the context of Theorem 3, the matrices $G_I^T H_J$ and $H_J^T G_I$ are submatrices of a matrix $M := \left[\langle g_i, h_j\rangle\right]$ and its transpose, respectively.*

Proof. Let $M_{J,I}$ denote the submatrix of M with column indices in J and row indices in I. The formula for matrix multiplication shows that $\left(G_I^T H_J\right)_{ij} = \left\langle G_{\{I_i\}}, H_{\{J_j\}}\right\rangle$ and it is easily seen that $(M_{J,I})_{ij} = \left\langle g_{I_i}, h_{J_j}\right\rangle$ which amounts to the same thing. Thus the matrices $G_I^T H_J$ and $M_{J,I}$ are equal. The matrix $H_J^T G_I$ is then equal to $\left(G_I^T H_J\right)^T = \left(M^T\right)_{I,J}$. □

Putting this all together, we propose the following adaptation of Algorithm 1.

Algorithm 2 Compute the angular spectrum of (P, Q)

Input: Sets G and H that generate $P, Q \in \mathcal{C}(\mathbb{R}^n)$ as in Theorem 3.
Output: The set $\Gamma(P, Q)$ of all critical angles of (P, Q).

```
Γ ← ∅
Compute the matrix M from Proposition 6
μ ← min_{i,j} (M_ij)
if μ ≥ 0 then
    Γ ← Γ ∪ {arccos (μ)} // Nonobtuse case, the maximal angle is critical
end if
if P ∩ −Q ≠ {0} then
    Γ ← Γ ∪ {π} // by Proposition 1
end if
for all I ∈ 𝓘(G) and J ∈ 𝓘(H) do
    G_I^T H_J ← M_{J,I} // Proposition 6
    H_J^T G_I ← (M^T)_{I,J}
    if π/2 ∉ Γ then
        Use Proposition 4 to find the eigenspace W_{π/2} for cos (θ) = 0
    end if
    Find all eigenspaces W_θ for cos (θ) ≠ 0 using Proposition 5
    for all eigenspaces W_θ do
        if θ ∈ Γ then
            Skip to the next W_θ
        else if θ = π then
            Skip to the next W_θ // We checked for π ∈ Γ (P, Q) earlier
        else if μ ≥ 0 and cos (θ) ≤ μ then
            Skip to the next W_θ // Nonobtuse case, μ is a true minimum
        else if any (ξ, η)^T ∈ W_θ satisfy (3) then
            Γ ← Γ ∪ {θ}
        end if
    end for
end for
return Γ
```

Unfortunately, we still expect to encounter eigenspaces of dimension greater than one in Algorithm 2. A partial solution to that problem is investigated forthwith, as we limit our attention to the maximal angle. This allows us to bring Theorem 1 and Corollary 2 to bear, eliminating at the outset any eigenspaces that correspond to nonobtuse angles.

6 Finding the maximal angle

In this section, we address the problem of finding only the maximal angle between two cones. The main benefit of doing so is that several optimizations become available as a consequence of our earlier results. These shortcuts offer not only efficiency improvements, but also avoid some of the large eigenspaces that plague Algorithm 2. The intention is that, if all such eigenspaces can be ruled out, then we can be sure that the algorithm works for the given cones.

Recall Example 5, which showed that large eigenspaces occur when the matrices G_I and H_J are square. This is one situation that can be avoided entirely during a maximal angle search.

Proposition 7. *If* $\operatorname{card}(I) = n$ *or* $\operatorname{card}(J) = n$ *in the context of Theorem 3, then the only eigenvalues satisfying Equation (4) are* $\cos(\theta) \in \{-1, 0, 1\}$.

Proof. The eigenvalue $\cos(\theta) = 0$ is indeed a possibility, so record it and suppose that $\cos(\theta) \neq 0$. If $\operatorname{card}(I) = n$, then G_I is square and its columns are linearly-independent. Substitute $G_I^+ = G_I^{-1}$ into Item 3 in Proposition 5 to obtain

$$H_J^+ H_J \eta = (\cos(\theta))^2 \eta.$$

The matrix $H_J^+ H_J$ is a projector by Proposition 2, and therefore has one nonzero eigenvalue $(\cos(\theta))^2 = 1$. Solve to obtain $\cos(\theta) \in \{-1, 1\}$. When $\operatorname{card}(J) = n$, the situation is analogous. ☐

A similar optimization can likely be obtained from Seeger and Sossa's Theorem 2.3 [24]. This result comes into play for the maximal angle because Theorem 1 can be used to rule out $\cos(\theta) \in [0, 1]$, and Proposition 1 rules out $\cos(\theta) = -1$. Having done so, we can ignore any index sets of cardinality n and the associated eigenspaces.

The other improvements that we make relative to Algorithm 2 are similar in spirit. Given generators for the cones P and Q, we first check whether or not $P \subseteq Q^*$ in Theorem 1. This can be done cheaply according to Proposition 6. If $\Theta(P, Q)$ is nonobtuse, then Corollary 2 tells us how to find it. Otherwise we proceed along the same lines, but skipping any angles (and their eigenspaces) that are too small to be maximal. This is in contrast to a critical angle search where even if the maximal angle is known, smaller angles may still be critical.

Algorithm 3 Compute the maximal angle between P and Q

Input: Sets G and H that generate $P, Q \in \mathcal{C}(\mathbb{R}^n)$ as in Theorem 3.
Output: The maximal angle $\Theta(P,Q)$ between P and Q.

```
Compute the matrix M from Proposition 6
μ ← min_{i,j} (M_ij)
if μ ≥ 0 then
    return arccos(μ) // by Theorem 1 and Corollary 2
end if
// Proceed assuming Θ(P,Q) ≥ arccos(μ)
if P ∩ −Q ≠ {0} then
    return π // by Proposition 1
end if
// Proceed assuming Θ(P,Q) ≠ π
for all I ∈ 𝓘(G) and J ∈ 𝓘(H) with cardinality < n  do // Proposition 7
    G_I^T H_J ← M_{J,I} // Proposition 6
    H_J^T G_I ← (M^T)_{I,J}
    Find all eigenspaces W_θ for cos(θ) ≠ 0 using Proposition 5
    for all eigenspaces W_θ do
        if cos(θ) = −1 or cos(θ) ≥ μ then
            Skip to the next W_θ  // θ has been ruled out or is too small
        else if any (ξ, η)^T ∈ W_θ satisfy (3) then
            μ ← cos(θ)
        end if
    end for
end for
return arccos(μ)
```

7 Conclusions

Algorithm 3 is still not a complete solution, but in some simple cases it can provide reassurance that the maximal angle has indeed been found.

Example 6 (Seeger and Sossa [24], Example 7.4)**.** Recall the standard basis $\{e_1, e_2, e_3, e_4, e_5\}$ in $\mathbb{R}^5$ and define

$$
\begin{aligned}
h_i &:= e_i - e_{i+1}, \\
P &:= \operatorname{cone}(\{e_1, e_2, e_3, e_4, e_5\}) = \mathbb{R}^5_+, \\
Q &:= \operatorname{cone}(\{h_1, h_2, h_3, h_4\}).
\end{aligned}
$$

Seeger and Sossa find $\Theta(P,Q)$ to be approximately 0.8524π. This result is confirmed by Algorithm 3: no eigenspaces of dimension greater than one are encountered. However, if we use Algorithm 2 to find the critical angles, then we run into trouble. There is an eigenspace of dimension two corresponding to the eigenvalue $\cos(\theta) = \sqrt{2}/2$. Is $\arccos\left(\sqrt{2}/2\right)$ a critical angle of (P,Q)?

We leave open the general question of whether or not either algorithm can be improved further. Theorem 5.2 of Seeger and Sossa [24] shows that the map $\Theta(\cdot,\cdot)$ is Lipschitz continuous: if P_2 is close to P_1 and if Q_2 is close to Q_1, then $\Theta(P_2, Q_2)$ is close to $\Theta(P_1, Q_1)$. One line of inquiry would be to perturb the cones P and Q, and to see if this can be done in a beneficial way.

Question 1. If $P = \operatorname{cone}(G)$ and $Q = \operatorname{cone}(H)$, how can we perturb the generators G and H to obtain new cones that are close to P and Q?

Question 2. Assuming we can answer Question 1, how do those perturbations affect the eigenspaces that appear in Proposition 3?

Taken together, these questions are harder than they look. For example, randomly perturbing the generators $G \coloneqq \{e_1, e_2, -e_2\}$ of the right half-plane in $\mathbb{R}^2$ might (with high probability) destroy the structure in the matrices G_I. But, that same perturbation (if it nudges e_2 or $-e_2$ to the left) can turn the right half-plane into the entire ambient space $\mathbb{R}^2$, which is not even in $\mathcal{C}(\mathbb{R}^2)$.

If we can eliminate the large eigenspaces with perturbations that don't change P and Q too much, then $\Theta(P, Q)$ should be close to the maximal angle that Algorithm 3 finds between the perturbed cones. In any case, a more holistic approach is needed if we are to tackle larger problems. In higher dimensions, the need for numerical linear algebra makes the dimension of an eigenspace a somewhat subjective matter. Under those circumstances, a list of special cases cannot simply be enumerated.

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