Introduction to Koecher Cones

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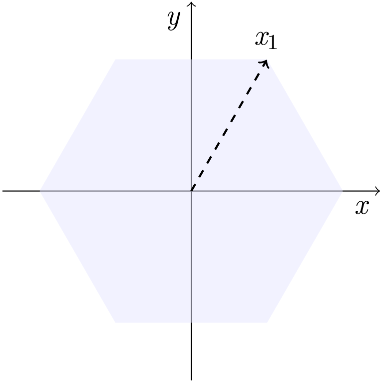
Definition 1. Let S be a convex set in some vector space. We say that the point $x \in S$ is an *extreme point* of the set S if $x = \lambda x_1 + (1 - \lambda)x_2$ for $\lambda \in (0, 1)$ implies $x_1 = x_2 = x$.

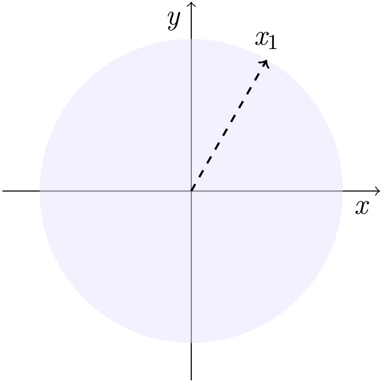
Intuitively, an extreme point is a "vertex" of S. For polyhedral sets, the intuition is accurate: the extreme points are simply the vertices.

For non-polyhedral convex sets, however, "vertices" can occur where the boundary of the set is rounded.

Example 2. Every vertex of a hexagon in $\mathbb{R}^2$ is an extreme point of the hexagon (which is a convex set).



Example 3. Every point on the boundary of a circle in $\mathbb{R}^2$ is an extreme point of the circle (which is a convex set).

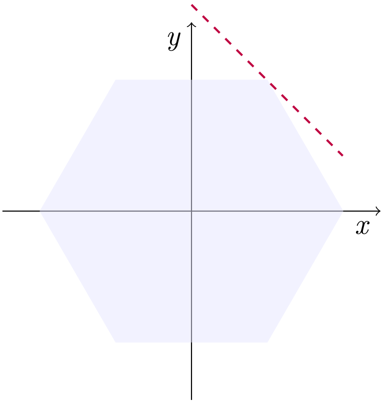


Exposed Points

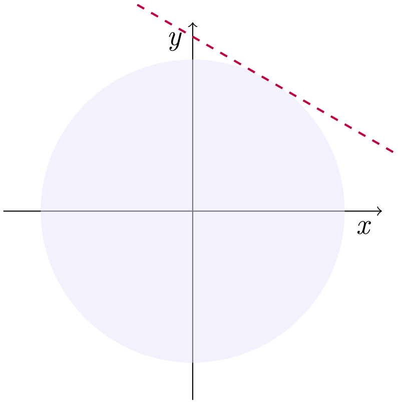
Definition 4. Let S be a convex set in some vector space. We say that the point $x \in S$ is an *exposed point* of the set S if there exists a hyperplane H with $H \cap S = x$.

Exposed points are similar, but not equivalent to, extreme points. For e.g. polyhedral sets in $\mathbb{R}^n$, the two concepts are the same.

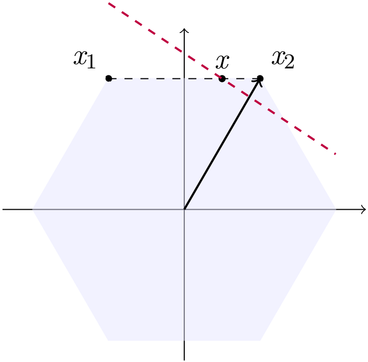
Example 5. Every vertex of a hexagon in $\mathbb{R}^2$ is an exposed point of the hexagon (which is a convex set).



Example 6. Every point on the boundary of a circle in $\mathbb{R}^2$ is an exposed point of the circle (which is a convex set).



Lemma 7. Suppose S is a convex set with non-empty interior. Then the exposed points of S are extreme points of S.



Proof.

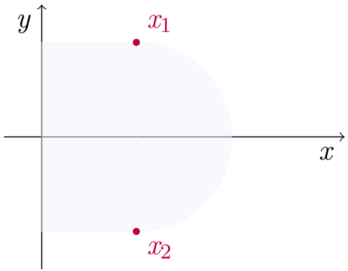
For simplicity, translate everything so that $0 \in \operatorname{int}(S) \neq \emptyset$. Now, suppose $x \in S$ is not an extreme point; then $x = \lambda x_1 + (1 - \lambda)x_2$ with $\lambda \in (0, 1)$ and $x_1 \neq x_2$.

Let H be a hyperplane containing x. If H contains one of $0, x_1, x_2$, then $H \cap S \neq \{x\}$ and we are done.

If H contains neither x_1 nor x_2, then H separates x_1 and x_2. Any hyperplane divides the space in half, so zero must belong to either the x_1 half or the x_2 half.

Without loss of generality, let 0 and x_1 lie in the same half-space. Then the segment $[0, x_2] \subset S$ must cross H at some point; that point lies in both H and S. Thus, $H \cap S \neq \{x\}$. $\square$

Example 8 (extreme points which are not exposed).



Extreme: no points on the right to add to a point on the left.

Not exposed: the only hyperplane that could work for the circle at x_1, x_2 is the (horizontal) tangent.

The concept of *extreme point* is not quite right when working with cones. The problem is that, if we have a convex set S with extreme point x, then in $\operatorname{cone}(S)$, the points along the edge λx $(\lambda > 0)$ are no longer extreme points.

This motivates the following:

Definition 9. Let K be a pointed, closed, and convex cone. We say that the direction $d \in K$ is an *extreme direction* of K if $d = \lambda_1 d_1 + \lambda_2 d_2$ for $\lambda_1, \lambda_2 > 0$ implies $d_1 \equiv d_2 \equiv d$.

Note the symmetry between this and the definition of *extreme point*. We've replaced "point" with "direction" and the convex combination with a conic combination. Two directions are considered the same if one is a scalar multiple of the other.

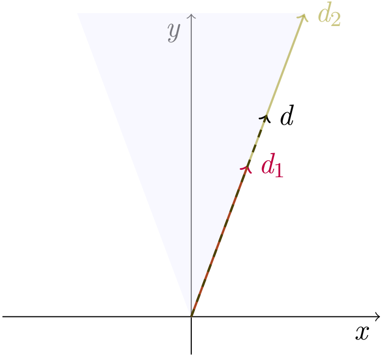
Some notation (beware the capitalization!):

Definition 10. The set of all extreme directions of K is denoted $\operatorname{Ext}(K)$.

Definition 11. The set of all extreme directions of K **with unit norm** is denoted $\operatorname{ext}(K)$.

This latter definition is useful because we don't have e.g. both $d \equiv 2d$ in $\operatorname{ext}(K)$.

Example 12. The direction d is clearly a convex combination of d_1 and d_2, but it is still an extreme direction, because we consider $d \equiv \lambda d_1 \equiv \mu d_2$ to be the same directions.

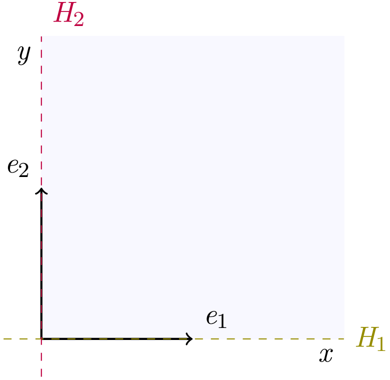


Exposed Directions

Likewise, we can replace our *exposed points* with something more appropriate for cones. The reasoning is the same: if we have a convex set S with exposed point x, then in $\operatorname{cone}(S)$, the points along the edge λx $(\lambda > 0)$ are no longer exposed.

Definition 13. Let K be a pointed, closed, and convex cone. We say that the direction $d \in K$ is an *exposed direction* of K if $F = \{\lambda d \mid \lambda > 0\}$ is an exposed face. That is, there exists a hyperplane H with $H \cap K = F$

Example 14. In $\mathbb{R}^2_+$, both of the standard basis vectors are exposed *directions*, even though they are not exposed *points*.



Theorem 15 (Carathéodory's Theorem). Let $S \subseteq \mathbb{R}^n$. Then every point in the convex hull of S can be expressed as a convex combination of at most $n+1$ points.

Proof (sketch). Suppose there are $k > n+1$ extreme points of S – otherwise, there's nothing to prove. Choose one point, say, x_1, to translate to the origin. The remaining nonzero vectors $(x_2 - x_1), (x_3 - x_1), \ldots, (x_k - x_1)$ are linearly-dependent, since there are more than n of them, and we're in $\mathbb{R}^n$.

We can use this freedom to choose λ_i such that $x = \sum_{i=1}^{k} \lambda_i x_i$ is a convex combination, and exactly one of the λ_i is zero. We throw out that term, leaving only $k-1$ points, and repeat the process until we can't anymore, when $k = n+1$.

Definition 16. A *conic combination* of $x_1, x_2, \ldots, x_n$ is a linear combination $\sum \alpha_i x_i$ where each coefficient $\alpha_i \geq 0$.

Theorem 17 (Carathéodory's Theorem for Cones). Let $S \subseteq \mathbb{R}^n$. Then every point in $\operatorname{cone}(S)$ can be expressed as a conic combination of at most n points in S.

Proof (idea). Similar to the previous theorem. To see why the theorem is true, you can imagine applying the convex hull version to a cross-section of the cone that lives in $\mathbb{R}^{n-1}$. You can then take the conic combination of the resulting n points.

Carathéodory

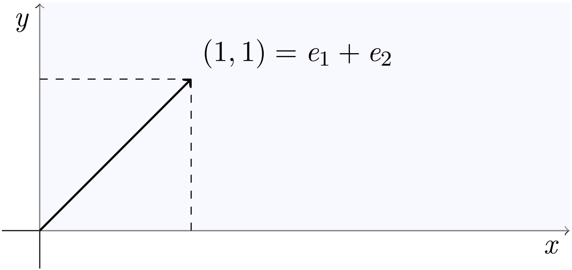
Inspired by Carathéodory's theorem, we make,

Definition 18. Let K be a proper cone. The *Carathéodory number* $\kappa(x)$ of the point $x \in K$ is the (minimum) number of extreme directions of K required to express x as a conic combination of said directions.

Definition 19. The *Carathéodory number* of K itself is,

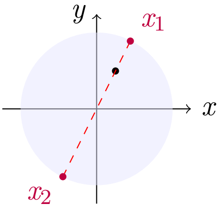
$$\kappa\left(K\right) = \max\left(\left\{\kappa(x) \mid x \in K\right\}\right)$$

Example 20 ($K = \mathbb{R}^2_+$). When K is the non-negative quadrant in $\mathbb{R}^2$, there are two extreme directions e_1 and e_2. The point $(1, 1)$ requires both extreme directions to be written as $(1, 1) = (1, 0) + (0, 1)$. Therefore, $\kappa(1, 1) = 2$. Moreover, any such point in K can be written as a conic combination of e_1 and e_2, so $\kappa(K) = 2$.



Example 21 ($K = \mathcal{L}^3_+$). Let K be the Lorentz "ice cream" cone in $\mathbb{R}^3$. At every height z, the cross-section of K is a closed disk. The boundary of this disk consists of (only) extreme directions of K.

We can express any point in this disk as a conic combination of two boundary points, and obviously one boundary point won't always work. Thus, $\kappa(\mathcal{L}^3_+) = 2$.



Example 22. $\kappa\left(\mathbb{R}^n_+\right) = n$ by analogy with $\mathbb{R}^2$, and clearly $\kappa(x) = \kappa\left(\mathbb{R}^n_+\right) = n$ for all $x \in \operatorname{int}\left(\mathbb{R}^n_+\right)$.

Can we extend the same idea to any polyhedral cone in $\mathbb{R}^n$? Sort of.

Suppose K is polyhedral with nonempty interior and let $S = \operatorname{Ext}(K)$. Then by Carathéodory's cone theorem, any $x \in \operatorname{cone}(S) = K$ can be expressed as a conic combination of at most n points of $S = \operatorname{Ext}(K)$, so $\kappa(K) \leq n$.

Can we also show $\kappa(K) \geq n$? Probably, but it needs a careful proof.

Homogeneity

Definition 23. We say that the cone K is *homogenous* if, for all x and y in the interior of K, there exists an automorphism of K sending x to y.

Equivalently, if $\operatorname{Aut}(K)$ represents the automorphism group of K, then K is homogenous if $\operatorname{Aut}(K)$ acts transitively on $\operatorname{int}(K)$:

$$\{Ax \mid A \in \operatorname{Aut}(K)\} = \operatorname{int}(K), \text{ for all } x \in \operatorname{int}(K)$$

(That is, $\operatorname{Aut}(K)$ has a single orbit.)

Proposition (2.4 [1]). Polyhedral K is homogenous if and only if $\operatorname{card}(\operatorname{Ext}(K)) = n$.

Proposition (2.1 [1]). Let K be a pointed, closed, convex cone and $A \in \operatorname{Aut}(K)$. Then $v \in \operatorname{ext}(K) \iff A(v) \in \operatorname{ext}(K)$.

Proof.

Since $\operatorname{Aut}(K)$ is a group, $A \in \operatorname{Aut}(K) \iff A^{-1} \in \operatorname{Aut}(K)$. Suppose $v \in \operatorname{ext}(K)$ but not $\frac{A(v)}{\|A(v)\|} \in \operatorname{ext}(K)$, i.e.,

$$A(v) = \lambda_1 w_1 + \lambda_2 w_2;\ \lambda_1, \lambda_2 > 0$$

Inverting, we get a contradiction:

$$v = \lambda_1 A^{-1}(w_1) + \lambda_2 A^{-1}(w_2);\ A^{-1}(w_i) \in K$$

$\square$

Theorem (4.3 [4]). Let K be a proper cone, and suppose $A \in \operatorname{Aut}(K)$. Then $v \in \operatorname{Ext}(K)$ is an exposed direction if and only if $A(v) \in \operatorname{Ext}(K)$ is an exposed direction.

Proof. Let $v \in \operatorname{Ext}(K)$ be an exposed direction. Then there exists a hyperplane H so that $H \cap K = \mathbb{R}_+ v$. If we apply A to both sides,

$$A(H \cap K) = A(\mathbb{R}_+ v) = \mathbb{R}_+ A(v)$$

By the previous proposition, $A(v) \in \operatorname{Ext}(K)$. So we would like to show that $A(H \cap K) = A(H) \cap K$ and that $A(H)$ is a hyperplane.

Homogeneity

Proof ($A(H \cap K) = A(H) \cap K$).

$$A(H \cap K) = \{A(x) \mid x \in H \text{ and } x \in K\}$$

Letting $A(x) = y \iff x = A^{-1}(y)$,

$$\begin{aligned} A(H \cap K) &= \left\{ y : A^{-1}(y) \in H \text{ and } A^{-1}(y) \in K \right\} \\ &= \{ y : y \in A(H) \text{ and } y \in A(K) \} \\ &= \{ y : y \in A(H) \text{ and } y \in K \} \\ &= A(H) \cap K \end{aligned}$$

Proof ($A(H)$ is a hyperplane). Suppose we define H by,

$$H \coloneqq \{x \in \mathbb{R}^n \mid \langle a, x \rangle = \alpha\}$$

Then,

$$\begin{aligned} A(H) &= \{A(x) \in \mathbb{R}^n \mid \langle a, x \rangle = \alpha\} \\ &= \left\{ y \in \mathbb{R}^n \;\middle|\; \langle a, A^{-1}(y) \rangle = \alpha \right\} \\ &= \left\{ y \in \mathbb{R}^n \;\middle|\; \langle \left(A^{-1}\right)^{*}(a), y \rangle = \alpha \right\} \\ &= \text{another hyperplane} \end{aligned}$$

Proposition (2.2 [1]). Suppose K is a homogenous cone. Then $\kappa(x) = \kappa(K)$ for all $x \in \operatorname{int}(K)$.

Proof.

Suppose $x \in \operatorname{int}(K)$ maximizes κ over that domain (some x must, by definition). Then, $\exists x_i \in \operatorname{Ext}(K)$ such that,

$$x = \sum_{i=1}^{\kappa(K)} \alpha_i x_i \iff y = A(x) = \sum_{i=1}^{\kappa(K)} \alpha_i A(x_i)$$

By the previous proposition, $A(x_i) \in \operatorname{Ext}(K)$, and since K is homogenous, every $y \in \operatorname{int}(K)$ arises in this manner. $\square$

Homogeneity

At this point it is natural to ask if the converse of the previous proposition holds. That is,

Question. Suppose $\kappa(x) = \kappa(K)$ for all $x \in \operatorname{int}(K)$. Does that imply that K is homogenous?

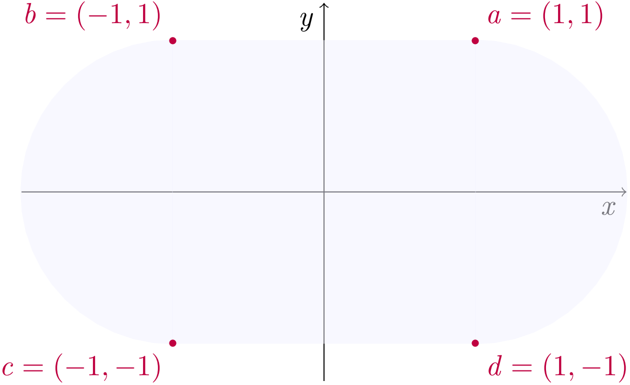
Tunçel and Xu [4] resolve this question in the negative with a counterexample:

$$S := \left\{ x \in \mathbb{R}^2 \;\middle|\; -1 \leq x_2 \leq 1, -1 - \sqrt{1 - x_2^2} \leq x_1 \leq 1 + \sqrt{1 - x_2^2} \right\}$$

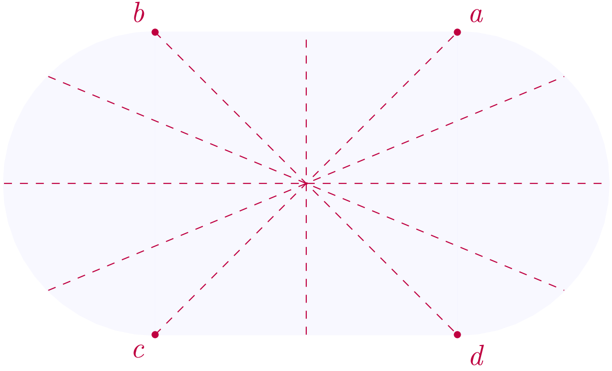
$$K := \left\{ \left(t, tx_1, tx_2\right)^T \;\middle|\; t \geq 0, x \in S \right\}$$

Homogeneity

The set S should look familiar:



And it shouldn't be hard to convince you that $\kappa(x) = \kappa(K) = 2$ for all $x \in \operatorname{int}(K)$:



Theorem 24. K is not homogenous [4].

Proof (idea). If K is homogenous, then there exists an automorphism of K that maps any $x \in \operatorname{int}(K)$ to any $y \in \operatorname{int}(K)$.

Note that $y = (1, 0, 0)^T$ lies in the interior of K.

To show that K is not homogenous, we seek a point $x = (1, x_1, x_2)^T \in \operatorname{int}(K)$ such that no automorphism of K sends x to $y = (1, 0, 0)^T$.

Remark (for the lazy). Kaneyuki and Tsuji [2] explicitly classified all homogenous convex cones in $\mathbb{R}^3$, and our K isn't one of them.

Proof. By our previous propositions and theorems, any automorphism of K will preserve extreme and exposed directions. Thus, for $A \in \operatorname{Aut}(K)$, we have,

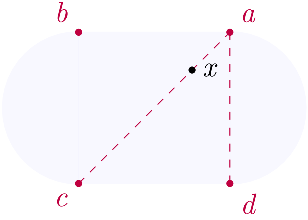
$$A(\{a, b, c, d\}) \equiv \{a, b, c, d\} \text{ (as directions)}$$

We consider the first of four cases, where $A(a) = a$; the rest are identical and follow by symmetry.

Proof (continued). Suppose we have a point $x \in \operatorname{int}(K)$ such that $x = \alpha \cdot a + \gamma \cdot c$ is a conic combination of a and c.

Then, $A(x) = \alpha A(a) + \gamma A(c) = \alpha a + \gamma A(c)$.

For $A(x) \in \operatorname{int}(K)$, we require $A(c) \equiv c$ or $A(c) \equiv d$ (as directions). Look at the picture until you believe it. The same holds for $A(d)$.



Proof (continued). Since we know what A does to a, c, d, we know that $A(b) \equiv b$. Let,

$$x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \in \operatorname{int}(K), \; A(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and note,

$$\begin{aligned} A^{-1}(a) &= A^{-1}\left(1,1,1\right)^T = \mu_1 a \\ A^{-1}(b) &= A^{-1}\left(1,-1,1\right)^T = \mu_2 b \end{aligned}$$

Proof (continued).

These imply,

$$A^{-1}\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right) = A^{-1}(c) = \begin{bmatrix}2-\mu_1\\2x_1-\mu_1\\2x_2-\mu_1\end{bmatrix} = \lambda\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \lambda c$$

Clearly we can choose x_1 and x_2 so that $x \in \operatorname{int}(K)$ and the last two equations above are inconsistent. Furthermore, we can find x_1, x_2 that additionally make $A^{-1}c = \lambda d$ impossible.

Therefore, no such A exists.

Koecher Cones (Finally)

In 1957, Koecher exhibited a family of cones which are self-dual but not homogenous [3]. Let $\rho \in (0,1)$, then define,

$$K_\rho := \operatorname{cl}\left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3 : u > 0, v > 0, |w| < u^\rho v^{1-\rho} \right\}$$

$$\alpha := \frac{1}{\rho}\left(\frac{\rho}{1-\rho}\right)^{1-\rho}$$

$$S := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Under the weighted inner product $\langle x, y \rangle_S = \langle Sx, y \rangle_{\mathbb{R}^3}$, the cone K is self-dual. Moreover, $\kappa(x) = 2$ for all $x \in \operatorname{int}(K)$.

This is interesting in the context of the previous propositions and theorems because it provides another counterexample to the question,

Question. Suppose $\kappa(x) = \kappa(K)$ for all $x \in \operatorname{int}(K)$. Does that imply that K is homogenous?

Moreover, it provides a *self-dual* counterexample. The previous counterexample was not self-dual.

Koecher Cones

This is interesting *for us* because we are interested in determining the possible Lyapunov ranks that a cone may possess.

Polyhedral cones, Lorentz cones, and a few other types are settled. Since many of the common cones have known Lyapunov ranks, we need to start looking at “weird” cones if we’re going to uncover one with an unexpected Lyapunov rank.

Cones constructed as counterexamples are good candidates.

> Convexity has an immensely rich structure and numerous applications. On the other hand, almost every "convex" idea can be explained by a two-dimensional picture.
>
> — Alexander Barvinok, *A Course in Convexity*

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